Note on Kovacic's algorithm*

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Abstract

There exists algorithms to find Liouvillian solutions of second order homogeneous differential equations (see [5, 9]). In this paper, we note that, by carefully combining the techniques of those algorithms, one can find the liouvillian zeroes of an irreducible second order linear differential equation by computing only rational solutions of some associated linear differential equations. If the equation is reducible, then one can either factor the equation or prove that the Galois group is completely reducible by also computing only rational solutions of an associated linear differential equation. The result is an easy-to-implement reformulation of the Kovacic algorithm for second order equations where in the irreducible case no algebraic extensions of the field of constants of the coefficient field is necessary.

1 Introduction

Let k be a differential field with derivation δ denoted $\delta y = y'$, $\delta^2 y = y''$,... The field of constants $\{c \in k | c' = 0\}$ is denoted $\mathcal{C}_k$. Let $L(y) = 0$ be a linear differential equation with coefficients in k. We consider 3 classes of solutions of $L(y) = 0$: the rational, resp. exponential, resp. Liouvillian solutions/zeros (over k) are the solutions of $L(y) = 0$ are those which are in k, resp. whose logarithmic derivative belong to k, resp. which can be written using $\int$, exponentials and algebraic functions (cf. [5, 6, 9]). We assume in the following that, when needed, differential fields come equipped with two algorithms: one that finds exponential zeroes and one that finds rational zeroes of linear operators (see e.g. [1, 2, 7]).

The existing algorithms [5, 6, 9] which compute Liouvillian solutions of $L(y) = 0$ are based on differential Galois theory and assume that the coefficients $L(y) = 0$ belong to a differential field K whose field of constants $\mathcal{C}_K$ is

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algebraically closed of characteristic 0. In this paper we always assume that such a field K exists and contains k (e.g. $k = \mathbb{Q}(x) \subset K = \overline{\mathbb{Q}}(x)$).

Theorem 1 (cf. [6], Theorem 2.4) *If $L(y) = 0$ with coefficients in K has a Liouvillian solution, then there is a solution z whose logarithmic derivative z'/z (a solution of the associated Riccati equation) is algebraic over K of degree bounded by a function depending only on the order of $L(y) = 0$ (not on the particular equation).*

The output of the algorithms is a minimal polynomial $P(u)$ of an algebraic logarithmic derivative z'/z. One proceeds in two steps:

1. Bound the degree of $P(u)$ (cf. [5, 6, 9]).
2. Compute the coefficients of $P(u)$.

In general, the computation of the coefficients of $P(u)$ is based on the computation of exponential solutions of some associated differential equations. The algorithm presented in this note is an easy-to-implement reformulation of Kovacic's algorithm for second order equations presented in [5]; in the sequel, the computation of exponential solutions is as much as possible (always for the irreducible case) replaced by the usually easier computation of rational solutions. For $k = \mathbb{Q}(\lambda)(x) \subset K = \overline{\mathbb{Q}}(x)$ one also has that the computation of rational solution can be done over k, i.e. no new algebraic constants need to be introduced (cf. [2] for details and further references).

2 Algebraic degree of a solution of the Riccati equation

In the algorithms [5, 6] one computes the minimal polynomial of an algebraic solution of *minimal algebraic degree* over K of the Riccati equation associated with a second order linear differential equation. In this note we show that there are some advantages in also considering algebraic solutions of the Riccati equation which are not of minimal degree.

We will use the fact that to a linear differential equation $L(y) = 0$ of order n and with coefficients in K one can associate a differential Galois group $\mathcal{G}(L)$ which is a linear algebraic group with a faithful action on the $\mathcal{C}_K$-vector space of solutions $\mathcal{V}(L)$ of $L(y) = 0$. Thus, after choosing a basis of $\mathcal{V}(L)$, we have $\mathcal{G}(L) \subseteq GL(n, \mathcal{C}_K)$.

Theorem 2 ([4], p. 41) *The differential Galois group $\mathcal{G}(L)$ of a differential equation of the form*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0 \qquad (a_i \in K)$$

is a unimodular group (i.e. $\mathcal{G}(L) \subseteq SL(n, \mathcal{C}_K)$) if and only if $\exists W \in K$, such that $W'/W = a_{n-1}$.

Using the variable transformation $y = z \cdot exp\left(-\frac{\int a_{n-1}}{n}\right)$ it is always possible to transform a given differential equation into one where the coefficient of $y^{(n-1)}$ is zero and thus whose differential Galois group is unimodular. The transformation does not alter the Liouvillian character of a solution. This will allow us to always assume, after transformation if necessary, that the differential Galois group is unimodular.

A group $G \subseteq GL(n, \mathcal{C}_K)$ is either reducible (i.e. has a non trivial invariant subspace), or imprimitive (i.e. $\mathcal{C}^n$ is a direct sum of non trivial subspaces that are permuted *transitively* by G, or primitive (i.e. irreducible but not imprimitive). Also, we say that L is completely reducible if it is the least common left multiple of a set of irreducible operators; in the second order case, this is equivalent to say that the Galois group admits a basis of eigenvectors. According to that, the linear algebraic subgroups of $SL(2, \mathcal{C}_K)$ can be classified as follows (cf. [5, 9]:

1. The reducible but not completely reducible groups (/i.e. the non-trivial G-invariant subspace has no complementary G-invariant subspace)
2. The completely reducible groups, in which the group is one of the following:
   (a) A reducible linear algebraic subgroups of $SL(2, \mathcal{C}_K)$ (i.e. a diagonal group)
   (b) An imprimitive subgroups of $SL(2, \mathcal{C}_K)$ which is either a finite groups $D_n^{SL_2}$ of order $4n$ (a central extensions of the dihedral groups D_n) and generated by:
   $$\begin{pmatrix} e^{\frac{\pi i}{n}} & 0 \\ 0 & e^{-\frac{\pi i}{n}} \end{pmatrix} \text{and} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$
   or the infinite group $D_\infty = \mathcal{C}_K{}^* \rtimes \mathbb{Z}/2\mathbb{Z}$.
   (c) An primitive finite subgroups of $SL(2, \mathcal{C}_K)$ which is isomorphic to central extensions of one of the permutation groups A_4, S_4 or A_5 and which we denote respectively $A_4^{SL_2}$, $S_4^{SL_2}$ and $A_5^{SL_2}$.
   (d) The group $SL(2, \mathcal{C}_K)$.

The degree of the minimal polynomial of the logarithmic derivative z'/z of a solution z equals the index of the stabiliser H of the $\mathcal{C}_K$ spann of z (cf. [9], Lemma 3.1). The group H has a one dimensional invariant subspace. The following counting process was used in [14]:

Lemma 1 *Let $L(y) = 0$ be a second order equation over K whose differential Galois group G is a finite unimodular group. Let $Z(G)$ be the center of G. Then, the number of irreducible minimal polynomials of degree $m < [G : Z(G)]$ of algebraic solutions of the Riccati equation is equal to $2/m$ times the number of* *maximal cyclic subgroups (i.e. not contained in a cyclic subgroups) of index m of G. In particular, this number is always finite. All other zeroes of the Riccati are algebraic of degree* $[G : Z(G)]$.

Proof. - Suppose w is an algebraic solution of the Ricatti equation; then, the degree m of the minimum polynomial of w equals the index $[G : H_1]$ of the stabilizer $H_1 = Stab_G(w)$ of w in G. Let y_1 be a corresponding solution of L (i.e. $y_1' = vy_1$). The solution y_1 is an eigenvector for all elements of H_1. Note that $Stab_G(w)$ always contains $Z(G)$.
If $m < [G : Z(G)]$, then H_1 in not central. By Maschke's theorem, there are up to scalar multiples exactly two common eigenvectors y_1 and y_2 to all elements of H_1. It follows that H_1 is abelian and, as it is unimodular, it is cyclic. Note that $H_2 = Stab_G(y_2'/y_2)$ contains H_1 and is also a cyclic subgroup of G with y_2 as an eigenvector. There is (up to scalar multiples) a unique basis of eigenvectors for H_2 in which H_1 will also be diagonal. Up to scalar the basis must be $\{y_1, y_2\}$ and H_2 stabilizes y_1'/y_1. Thus $H_1 = H_2$ and H_1 is a maximal cyclic subgroup. In particular y_1'/y_1 and y_2'/y_2 have the same algebraic degree $m = [G : H_1]$ over K. Thus, each maximal cyclic subgroup of index m provides two different logarithmic derivatives which are algebraic of degree m. Gathering them by groups of m, we get the number of irreducible minimal polynomials of degree m of algebraic solutions of the Riccati equation. □

For later reference we summarise the numbers obtained by computation in the group theory system CAYLEY for some finite groups needed later:

Corollary 2 *Let* $L(y) = 0$ *be a second order equation over* K *whose differential Galois group* G *is an irreducible subgroup of* $SL(2, \mathcal{C}_K)$. *For the possible minimal polynomials of an algebraic solution of the Riccati equation we get:*

- *If* $G \cong D_2^{SL_2}$ *(i.e., G is the quaternion group), there are exactly three minimal polynomials of degree 2 and all the others are of degree 4.*
- *If* $G \cong A_4^{SL_2}$*, there are exactly two minimal polynomials of degree 4, one of degree 6, and all the others are of degree 12.*
- *If* $G \cong S_4^{SL_2}$*, there is exactly one minimal polynomial of degree 6, one of degree 8, one of degree 12, and all the others are of degree 24.*
- *If* $G \cong A_5^{SL_2}$*, there is exactly one minimal polynomial of degree 12, one of degree 20, one of degree 30, and all the others are of degree 60.*

Proof. - We exibit the reasoning for the Icosahedral group $A_5^{SL_2}$. There are 15 maximal cyclic subgroups of order 4 (index 30), 10 of order 6 (index 20) and 6 of order 10 (index 12). This gives $2 \cdot 15$ solutions of the Riccati equation associated with $L(y) = 0$ which are algebraic of degree 30, $2 \cdot 10$ which are algebraic of degree 20 and $2 \cdot 6$ which are algebraic of degree 12. Thus there exists exactly one minimal polynomial of an algebraic solution of the Riccati

equation of degree 30, 20 and 12. All other solutions of the Riccati are algebraic of degree 60.
The proof for the other groups is along the same lines. ∎

In the following we will need differential equations associated to $L(y) = 0$:

Definition 3 *Let $L(y) = 0$ be a linear differential equation of degree n and fundamental system of solutions $\{y_1, \cdots, y_n\}$; the differential equation $L^{\textcircled{s}m}(y)$ whose solution space is spanned by the homogeneous forms of degree m in $y_1, \cdots, y_n$ is called the m-th symmetric power of $L(y) = 0$.*

In [10] an algorithm to construct the equation $L^{\textcircled{s}m}(y)$, which is of order at most $\binom{n+m-1}{n-1}$, is given. For second order equations (i.e. $n = 2$) the order of $L^{\textcircled{s}m}(y)$ is exactly $m + 1$ and universal formulaes for $L^{\textcircled{s}m}(y)$ can be derived in this case (cf. [10], Lemma 3.5). In particular for second order equations the homorphism Φ_m given in [9] p. 42 is an isomorphism which gives a bijection between rational solution (resp. exponential solutions) of $L^{\textcircled{s}m}(y) = 0$ and invariants (resp. semi-invariants) of degree m of $\mathcal{G}(L) \subseteq GL(n, \mathcal{C}_K)$. The character χ_m of $\mathcal{G}(L^{\textcircled{s}m})$ can be computed from the character of $\mathcal{G}(L)$ according to the formula given in [10] p. 15.

The goal of the next section is to reformulate for arbitrary second order equations a result given in [5] for differential equations of the form $y'' - ry$ ($r \in \mathbb{C}(x)$): There is a bijection between polynomials $P(u)$ of degree m whose roots are solutions of the Riccati equation and exponential solutions of $L^{\textcircled{s}m}(y) = 0$ (i.e. semi-invariants of order m of $\mathcal{G}(L)$).

3 Computing the coefficients of the minimal polynomial

The main reason for the efficiency of the Kovacic algorithm is a recursion for the coefficients of the minimal polynomial $P(u)$ of an algebraic solution of the Riccati equation $Ri(u) := u' - r + u^2 = 0$. In this section, we briefly recall the recursion by showing how it applies to $Ri(u) := u' - l_0 - l_1 u + u^2 = 0$, the Riccati equation corresponding to $y'' + l_1 y' + l_0 y$.

We now introduce the property of P that induces the recursion. If we differentiate $P(u)$, we get $P(u)' = \partial_K P + u' \frac{\partial P}{\partial u} = \partial_K P + (l_0 + l_1 u - u^2) \frac{\partial P}{\partial u} + Ri(u) \frac{\partial P}{\partial u}$, where $\partial_K P$ denotes the polynomial obtained from P by taking the derivative of all the coefficients. Thus, at a zero of Ri, we can introduce the *derivative modulo Ri* as $\partial_{Ri} = \partial_K + (l_0 + l_1 u - u^2) \frac{\partial}{\partial u}$. Now, if P is the minimum polynomial for a solution of Ri, we have $P(u) = P(u)' = 0$, and thus $\partial_{Ri} P(u) = 0$. As $\partial_{Ri} P(u)$ is a polynomial in u, the minimality of P implies that P divides $\partial_{Ri} P$. We will say that a polynomial is *special* if P divides $\partial_{Ri} P$. This property is crucial, as shown in the next lemma:

Lemma 4 *The set of polynomials $P(u)$ such that all their zeroes are zeroes of the Ricatti equation is exactly the set of special polynomials.*

Proof. - Suppose P is special. It is easilly seen (see e.g [15]) that all its factors are themselves special (and, conversely, the product of two specials is special) so we may assume that P is irreducible. Now, a zero of P is a zero of $\partial_{Ri}P$; thus, if $P(u)$=0, we get that $0 = P(u)' = \partial_{Ri}P(u) + Ri(u)\frac{\partial P}{\partial u}$. As P is irreducible, it is prime with $\frac{\partial P}{\partial u}$, and so $Ri(u) = 0$. Conversely, if all zeroes of P are zeroes of the Ricatti equation, we get that all zeroes of P are zeroes of $\partial_{Ri}P$, and so P divides $\partial_{Ri}P$. $\square$

Now, pick a polynomial $P(U) = U^m + b_{m-1}U^{m-1} + \ldots + b_0$ with unknown coefficients and suppose it is special: then, dividing $\partial_{Ri}P$ by P, one gets that $\partial_{Ri}P = (-mu+ml_1+b_{m-1})P$. Writing down that the remainder of this division shows that P is special if and only if the b_i satisfy the following system (adapted from [5, 3]):

$$(\sharp)_m \begin{cases} b_m = +1 \\ b_{i-1} = \dfrac{-b_i' + b_{m-1}b_i - l_1(i-m)b_i - l_0(i+1)b_{i+1}}{m-i+1} & m \geq i \geq 0 \\ b_{-1} = 0 \end{cases} \qquad (1)$$

So, the b_i are all determined from the knowledge of a suitable b_{m-1}. Now, let $u_1, \cdots, u_m$ be the zeroes of P in some algebraic closure of K; then, $b_{m-1} = -(u_1 + \cdots + u_m) = -(\frac{y_1'}{y_1} + \cdots + \frac{y_m'}{y_m}) = -\frac{(y_1 \cdots y_m)'}{y_1 \cdots y_m}$, so b_{m-1} is the logarithmic derivative of a solution of $L^{\circledS m}(y) = 0$, which is the key to the following theorem, adapted from Kovacic:

Theorem 3 *(after [5]) Let $L(y) = y'' - l_1y' - l_0y$ be a second order equation with $l_i \in k$, then $P(U) = U^m + \sum_{i=0}^{m-1} b_iU^i$ is a (not necessary irreducible) polynomial with coefficients in K whose roots are all solution of the associated Riccati equation of $L(y) = 0$ if and only if b_{m-1} is an exponential solution (over K) of $L^{\circledS m}(y) = 0$ (i.e. $\exists z$, such that $L^{\circledS m}(z) = 0$ and $z'/z = b_{m-1} \in K$).*

Proof. - That b_{m-1} is the logarithmic derivative of a solution of $L^{\circledS m}(y) = 0$ is clear from the above, and from $(\sharp)_m$ we see that the coefficients b_i belong to K if and only if b_{m-1} does.
We now show that the logarithmic derivative of any solution of $L^{\circledS m}(y) = 0$ will give a special polynomial (over $\overline{K}$). The exponential solutions of $L^{\circledS m}(y) = 0$ will be those corresponding to special polynomials with coefficients in K. Consider a form of degree m in $y_1, \ldots, y_n$ over $\mathcal{C}_K$. Since this is also a homogeneous form in two variables in terms of a fundamental system of solutions of $L(y) = 0$, one can factor this form into linear factors over $\mathcal{C}_K$. Since the product of special polynomials is special, we only need to prove that the logarithmic

derivative u_1 of a forms of degree one, i.e. any zero of Ri, gives a special polynomial. But clearly $u - u_1$ is special, since $\partial_{Ri}(u - u_1) = -u_1' + l_0 + l_1 u - u^2 = (l_1 - u - u_1)(u - u_1)$. $\Box$

This gives a bijection between monic polynomials of degree m over K whose roots are solutions of the Riccati equation and exponential solutions of $L^{\circledS m}(y) = 0$, or in other terms between semi-invariants of degree m of $\mathcal{G}(L)$. In the next section we will look for special polynomials corresponding to invariants. The bijection no longer exists for higher order linear differential equations:

Example. - Let $L(y) = 0$ be a third order equation with $\mathcal{G}(L) \subseteq SL(3, \mathcal{C}_K)$ an irreducible 3 dimensional representation of A_4 (unique up to equivalence). An example of such an equation is $H^{\circledS 2}(y)$, where $H(y) = 0$ is a second order differential equation with $\mathcal{G}(H) \cong A_4^{SL_2}$ (e.g. [5], p. 23). Since A_4 has no subgroups of index 2, $\mathcal{G}(L)$ has no 1-reducible subgroups of index 2. Thus there are no minimal polynomials of degree 2 of algebraic solutions of the associated Riccati equation. But from the decomposition of the character of $\mathcal{G}\left(L^{\circledS 2}\right)$ (cf. [10]) which has 3 summand of degree one, one get that $L^{\circledS 2}(y) = 0$ has at least 2 non-trivial exponential solutions (cf. [10], Lemma 3.5) which do not correspond to the minimal polynomial of an exponential solution. $\diamond$

Remark. - For third order equation the minimum polynomial of an algebraic solution of the Riccati equation is no longer special, and so the previous construction does not hold any more. However, it is shown in [15] (section 5) that the relation still holds between the first coefficient of a monic special polynomial and the exponential solutions of $L^{\circledS m}(y) = 0$. $\diamond$

4 The algorithm

In this section we will always assume that $L(y)$ is a second order equation with a differential Galois group $\mathcal{G}(L)$ which is a subgroup of $SL(2, \mathcal{C}_K)$. The previous section shows that there is a bijection between exponential solutions of $L^{\circledS m}(y) = 0$ and polynomials of degree m whose zeros are solutions of the Riccati. We now propose an algorithm where rational solutions of $L^{\circledS m}(y) = 0$ are used as much as possible.

We summarize the result of this section in the following:

Theorem 4 *Let $L(y)$ be a second order equation with $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$, then*

1. *$L(y) = 0$ as a two dimensional rational solution space if and only if $\mathcal{G}(L)$ is the identity.*

2. *$L(y) = 0$ as a one dimensional rational solution space if and only if $\mathcal{G}(L)$ is conjugate to*

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathcal{C}_K \right\}$$

3. *If $L^{\circledS 2}(y) = 0$ has a non trivial rational solution, then $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is reducible. Assuming that $L(y)$ as no non trivial rational solution we get:*

   (a) *The rational solution space of $L^{\circledS 2}(y)$ is three dimensional if and only if $\mathcal{G}(L) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$. The special polynomial of order two associated with the logarithmic derivative of any rational solution of $L^{\circledS 2}(y) = 0$ is a square.*

   (b) *If the rational solution space of $L^{\circledS 2}(y)$ is one dimensional and the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{\circledS 2}(y) = 0$ is a square, then $\mathcal{G}(L)$ is congugated to a subgroup of*

   $$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid b \in \mathcal{C}_K, a = \pm 1 \right\}$$

   (c) *If the rational solution space of $L^{\circledS 2}(y)$ is one dimensional and the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{\circledS 2}(y) = 0$ factors over K but is not a square, then $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is completely reducible. A factorization, and thus a Liouvillian solution, of $L(y) = 0$ can be obtained by factoring the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{\circledS 2}(y) = 0$.*

4. *If $L^{\circledS 2}(y) = 0$ has no non trivial rational solution, then*

   (a) *If $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is reducible, then $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is not completely reducible. There will be a unique right factor of order one of $L(y) = 0$, and thus a unique rational solution of the Riccati.*

   (b) *If $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is irreducible, then*

      i. *If $L^{\circledS 4}$ has a rational solution q, then $\mathcal{G}(L)$ is an imprimitive subgroup of $SL(2, \mathcal{C}_K)$. For the special polynomial P_q obtained from the logarithmic derivative of q the following holds:*

         A. *If $L^{\circledS 4}$ has a one dimensional rational solution space, then P_q is the square of a unique special polynomial of order 2. In this case (L) is not the group of quaternions.*

         B. *If $L^{\circledS 4}$ has a two dimensional rational solution space, then P_q is either the square of a special polynomial of order 2 or it is irreducible. In this case $\mathcal{G}(L)$ is the quaternion group.*

      ii. *If $L^{\circledS 6}$ has a one dimensional rational solution space and $L^{\circledS 4}$ has no non-trivial rational solution, then the special polynomial obtained from the logarithmic derivative of a rational solution of $L^{\circledS 6}$ is irreducible. In this case $\mathcal{G}(L)$ is the tetrahedral group.*

iii. If $L^{ⓢ 8}$ has a one dimensional rational solution space and $L^{ⓢ 6}$ and $L^{ⓢ 4}$ have no non-trivial rational solution, then the special polynomial obtained from the logarithmic derivative of a rational solution of $L^{ⓢ 8}$ is irreducible. In this case $\mathcal{G}(L)$ is the Octahedral group.

iv. If $L^{ⓢ 12}$ has a one dimensional rational solution space and $L^{ⓢ 8}$, $L^{ⓢ 6}$ and $L^{ⓢ 4}$ have no non-trivial rational solution, then the special polynomial obtained from the logarithmic derivative of a rational solution of $L^{ⓢ 12}$ is irreducible. In this case $\mathcal{G}(L)$ is the Icosahedral group.

v. If $L^{ⓢ m}$ has no rational solution for $m \in \{4, 6, 8, 12\}$, then $\mathcal{G}(L) = SL(2, \mathcal{C}_K)$.

The proof of the above Theorem follows from the results of the following subsections. The proposed algorithm can be outlined as follows:

1. Compute the non trivial rational solutions of $L(y)$. This gives two Liouvillian solutions of $L(y)$ (e.g. by d'Alembert's method if only one rational solution is found).

2. Compute the a non trivial rational solutions of $L^{ⓢ 2}$ and test if the special polynomial of order two associated with the logarithmic derivative of a rational solution factors over K. If this is the case, this gives a factorization of $L(y)$ and two Liouvillian solutions (e.g. by d'Alembert's method if the special polynomial is a square). One only has to consider this case if the solution space of $L^{ⓢ 2}$ is either one or three dimensional (in which case the special polynomial always factors), so that at most one special polynomial has to be considered.

3. Test if $L(y)$ has a non trivial exponential solution. Such a solution must be unique and gives a factorization of $L(y)$ and (e.g. by d'Alembert's method) two Liouvillian solutions of $L(y)$.

4. Test if $L^{ⓢ 4}$ has non trivial rational solutions and conclude depending from the dimension of this rational solution space as stated in the theorem.

5. Test for increasing $m \in \{6, 8, 12\}$ if $L^{ⓢ m}$ as a non trivial rational solution and conclude as stated in the theorem.

6. Conclude that $L(y) = 0$ has no Liouvillian solution.

The steps have to be performed in the given order and the algorithm terminates as soon as a solution is found in one of the cases. The third step is the only one where instead of some rational solution one has to compute an exponential solution of $L(y)$ which is however known to be unique. This case corresponds to a reducible but non completely reducible group $\mathcal{G}(L) \subset SL(2, \mathcal{C}_K)$.

4.1 The reducible case

From [10] Proposition 4.2 one gets that if $L(y)$ has a rational solution, then

1. if $\mathcal{G}(L) \subset SL(2, \mathcal{C}_K)$ is completely reducible, then $L(y)$ will have two rational solution and thus $\mathcal{G}(L)$ is the identity.

2. if $\mathcal{G}(L) \subset SL(2, \mathcal{C}_K)$ is not completely reducible, then $\mathcal{G}(L)$ is conjugate to

$$\left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathcal{C}_K \right\}$$

Lemma 5 *Let $L(y)$ be a second order equation with $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ having no non trivial rational solutions, then*

1. *If the rational solution space of $L^{ⓢ2}(y)$ is one dimensional and the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{ⓢ2}(y) = 0$ is a square, then $\mathcal{G}(L)$ is conjugated to*

$$\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid b \in \mathcal{C}_K, a = \pm 1 \right\}$$

2. *If the rational solution space of $L^{ⓢ2}(y)$ is one dimensional and the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{ⓢ2}(y) = 0$ factors over K but is not a square, then $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is completely reducible. A factorization, and thus a Liouvillian solution, of $L(y) = 0$ can be obtained by factoring the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{ⓢ2}(y) = 0$.*

3. *If $L^{ⓢ2}(y) = 0$ has no non trivial rational solution and $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is reducible, then $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is not completely reducible. There will be a unique right factor of order one of $L(y) = 0$, and thus a unique rational solution of the Riccati.*

Proof. - We first note that if $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is irreducible (i.e. primitive or imprimitive), then $L^{ⓢ2}$ has no non trivial rational solution. This follows for the primitive case from [10], Proposition 2.4 and Table 1, and for the imprimitive case from the proof of the next section, since the smallest invariant is of degree 4 in this case. It follows that if $L^{ⓢ2}(y) = 0$ has a non trivial rational solution, then $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is reducible.
Also, if the special polynomial of order two associated with the logarithmic derivative of a rational solution of $L^{ⓢ2}$ factors into two distinct factors over K, then the Riccati equation has two distinct solution and thus $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ has a basis of eigenvectors and must be completely reducible.

Conversely, if $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is reducible and completely reducible, then there exists a basis $\{y_1, y_2\}$ of eigenvectors for $\mathcal{G}(L)$, i.e. $\forall \sigma \in \mathcal{G}(L), \sigma(y_1) = c_\sigma y_1$ and $\sigma(y_2) = c_\sigma^{-1} y_2$; thus, $y_1 y_2$ is an invariant of degree 2 of $\mathcal{G}(L)$.

- If this invariant is unique up to multiples, then the special polynomial of order two associated with its logarithmic derivative must be the product of the special polynomials associated with the logarithmic derivative of the exponential solutions y_1 and y_2. In particular, it can not be the square of a semi-invariant, i.e. the associated special polynomial is not a square.

- If this invariant of degree 2 is not unique. Then, computation in the basis $\{y_1, y_2\}$ shows immediately that $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ can have an invariant which is not a multiple of $y_1 y_2$ if and only if $\mathcal{G}$ consist only of scalar multiplication and thus, since $L(y)$ has no non trivial rational solution, $\mathcal{G}$ is of order 2. In this case any linear combination of y_1 and y_2 is a semi-invariant. Using this fact and the fact that a homogeneous form in two variables over an algebraic closed field factors into linear forms, we get that in this case the special polynomial of order two associated with the logarithmic derivative of any rational solution of $L^{\circledS 2}$ must factor.

If $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is not completely reducible and the special polynomial associated with a rational solution of $L^{\circledS 2}$ is a square, then the unique semi invariant of order one of $\mathcal{G}(L)$ has a character of order 2 and we get the result in this case from [10] Proposition 4.2. □

Remark. - The fact that factorization of differential operators is easier in the completely reducible case is used by Singer in [8]. The above results shows that by computing a non trivial rational solution q of $L^{\circledS 2}(y) = 0$ and factoring the associated special polynomial one either gets a right first order factor of $L(y)$ (if the special polynomial associated to the logarithmic derivative of q is a square) or, if the special polynomial factors, a proof that $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is completely reducible and reducible. ◇

Example. - Let $L(y) = y'' + \frac{3}{16x^2} y$. Computing the second symmetric power, we get

$$L^{\circledS 2}(y) = y''' + \frac{3}{4x^2} y' - \frac{3}{4x^3} y$$

This last equation has a one dimensional space of rational solutions generated by x. From the proposition, we get that $\mathcal{G}(L) \subseteq SL(2, \mathcal{C}_K)$ is a reducible group and that $L(y) = 0$ factors as a differential operator. The special polynomial obtained from the logarithmic derivative $1/x$ of x is

$$U^2 - \frac{1}{x} U + \frac{3}{16x^2}$$

which factors into $\left(U - \frac{1}{4x}\right)\left(U - \frac{3}{4x}\right)$. This gives two Liouvillian solutions of $L(y) = 0$:

$$y_1 = e^{\int \frac{1}{4x}} \quad , \quad y_2 = e^{\int \frac{3}{4x}}$$

The equation $L(y) = 0$ is the least common left multiple of $y' - \frac{1}{4x}y$ and $y' - \frac{3}{4x}y$. $\diamond$

Example. - The following example illustrates the fact that a quadratic extension of the constant field is sometimes necessary to factor the special polynomial obtained. Let now $L(y) = y'' + \frac{7}{16\,x^2}y$. Again, x is a generator of the one-dimensional space of rational solutions of $L^{\circledS 2}$ and we get the special polynomial

$$U^2 - \frac{U}{x} + \frac{7}{16\,x^2}$$

which factors into $\left(U - \frac{2 - i\sqrt{3}}{4x}\right)\left(U - \frac{2 + i\sqrt{3}}{4x}\right)$ (but does not factor on $\mathbb{Q}(x)$ which was our base field). This provides again two Liouvillian solutions of $L(y) = 0$:

$$y_1 = e^{\int \left(\frac{2 - i\sqrt{3}}{4x}\right)} \quad , \quad y_2 = e^{\int \left(\frac{2 + i\sqrt{3}}{4x}\right)}$$

Note that, in such a case, we can always avoid the algebraic number i (but not $\sqrt{3}$) by transforming the conjugated exponentials into sines and cosines; in our case, this gives the solutions $\sqrt{x}\cos(\frac{\sqrt{3}}{4}\log x)$ and $\sqrt{x}\sin(\frac{\sqrt{3}}{4}\log x)$ $\diamond$

Now, the two following examples show cases when the operator is reducible but not completely reducible (i.e, there is exactly one rational solution of the Ricatti equation).

Example. - Consider $L(y) = y'' - (1 + x^2)y$. One easily checks that $y_1 = e^{\frac{x^2}{2}}$ is a solution. Now, d'Alembert's substitution yields the other solution $y_2 = e^{\frac{x^2}{2}} \int e^{-x^2}$. So, if there was another rational solution to the Ricatti equation, $\int e^{-x^2}$ would be elementary, a contradiction. Of course, this can also be found by checking that $L^{\circledS 2}$ has no rational solution (and finding a unique exponential solution). $\diamond$

Example. - Consider $L(y) = y'' + \left(\frac{3}{16\,x^2} + \frac{1}{4\,(x-1)^2} - \frac{1}{4\,x\,(x-1)}\right)y$. There, $L^{\circledS 2}$ has no rational solution, but the Ricatti equation has the unique rational solution given by $4\,x\,(x-1)\,U + 1 - 3\,x = 0$. This yields the solution $y = x^{\frac{1}{4}}\sqrt{x - 1}$

for L.
Note that, here, $L^{\circledS 4}$ has a one-dimensional solution space generated by $x\,(x-1)^2$. This example illustrates the fact that there can be a unique solution to the Ricatti equation even if there are invariants (of degree higher than 2); also it shows that, to apply the techniques described in the next section, one must really test the reducibility of L. $\diamond$

4.2 The imprimitive case

In this case we show that the computation of a Liouvillian solution of a second order equation $L(y)=0$ is reduced to the computation of a rational solution of $L^{\circledS 4}$ and that the special polynomial associated to the logarithmic derivative is either a square or irreducible.

Lemma 6 *Let $L(y)=0$ be an irreducible second order equation over K whose galois group $\mathcal{G}(L)$ is unimodular. Then $\mathcal{G}(L)$ is an imprimitive subgroup of $SL_2(\mathcal{C}_K)$ if and only if $L^{\circledS 4}$ has a rational solution q. The special polynomial obtained from the logarithmic derivative of q is then*

1. *The square of a unique special polynomial of order 2 if $L^{\circledS 4}$ has a one dimensional rational solution space.*
2. *Either the square of a special polynomial of order 2 or is irreducible if $L^{\circledS 4}$ has a two dimensional rational solution space, in which case $\mathcal{G}(L) \cong D_2^{SL_2}$.*

Proof. - In [10] Theorem 4.1 it is shown that the differential Galois group of a second order linear differential equation with unimodular group is imprimitive if and only if $L^{\circledS 2}$ has an exponential solution whose square is in k, i.e. whose square is a rational solution of $L^{\circledS 4}$. Since $L^{\circledS 4}$ has no rational solution if $\mathcal{G}(L)$ is a finite primitive subgroups of $SL(2,\mathcal{C}_K)$ (cf. character decompositions in the next subsection), we get the fist assertion.
If the space of rational solutions of $L^{\circledS 4}$ is one dimensional, then, up to a constant, this rational solution is the square of the exponential solution of $L^{\circledS 2}$. Thus, the (unique) minimal polynomial corresponding to the (unique) logarithmic derivative of a rational solution of $L^{\circledS 4}$ will be the square of the minimal polynomial associated with the exponential solution of $L^{\circledS 2}$. Since there is a bijection between rational solutions and invariants, we look at the ring of invariants. The ring of invariants of $D_n^{SL_2}$ is generated by (cf. [12], p. 95)

$$I_1 = y_1^2 y_2^2,\, I_2 = y_1^{2n} + (-1)^n y_2^{2n},\, I_3 = y_1 y_2 (y_1^{2n} - (-1)^n y_2^{2n})$$

where $\{y_1, y_2\}$ is a basis of the representation module of $D_n^{SL_2}$ (i.e. of the solution space). While D_∞ has clearly only the invariant $y_1^2 y_2^2$ which must be the square of a unique semi-invariant. Thus, for $\mathcal{G}(L)$ not isomorphic to $D_2^{SL_2}$ (the group of quaternions), the special polynomial associated with a logarithmic derivative of a rational solution of $L^{\circledS 4}$ will be a square. For $D_2^{SL_2}$ the space of rational solutions of $L^{\circledS 4}$ is of dimension 2 (cf. also the character χ_4 given below). The group $D_2^{SL_2}$ has 4 irreducible characters, the trivial one denoted $\mathbf{1}$, 3 characters $\zeta_{1,1}$, $\zeta_{1,2}$, $\zeta_{1,3}$ of degree one and one character ζ_2 of degree two. The non trivial characters of degree one have the property that the product $\zeta_{1,i}\zeta_{1,j}$ is $\mathbf{1}$ for $i = j$ and different from $\mathbf{1}$ otherwise. If a second order equation $L(y) = 0$ has Galois group $\mathcal{G}(L) \cong D_2^{SL_2}$, then the corresponding character of $\mathcal{G}(L)$ will be ζ_2. The character χ_m of $\mathcal{G}(L^{\circledS m})$ can be computed according to the formula given in [10] p. 15:

$$\chi_2 = \zeta_{1,1} + \zeta_{1,2} + \zeta_{1,3}, \qquad \chi_3 = 2\zeta_2, \qquad \chi_4 = 2 \cdot \mathbf{1} + \zeta_{1,1} + \zeta_{1,2} + \zeta_{1,3}$$

this shows that there are 3 semi invariants I_i associated to the characters $\zeta_{1,i}$ ($i \in \{1, 2, 3\}$) whose square is rational. The products $I_1 I_2$, $I_1 I_3$ and $I_2 I_3$ are not invariants (i.e. do not correspond to a rational solution) since the products of the associated characters are not the trivial character. Thus a rational solution is either the square of a semi-invariant of order 2, in which case the special polynomial associated will be a square, or it is not the product of semi-invariants and the special polynomial associated will be irreducible. □

Example. - Consider the irreducible equation

$$L(y) = y'' - \left(-\frac{3}{16\,x^2} - \frac{2}{9\,(x-1)^2} + \frac{2}{9\,x\,(x-1)}\right) y = 0$$

The equation $L^{\circledS 4}$ has a one dimensional space of rational solutions generated by $x(x-1)^2$. The special polynomial associated with the logarithmic derivative $\dfrac{3x-1}{x^2-x}$ is

$$U^4 - \frac{3x-1}{x(x-1)} U^3 + \frac{243\,x^2 - 166\,x + 27}{72x^2(x-1)^2} U^2 - \frac{(3x-1)(81x^2-58x+9)}{144x^3(x-1)^3} U + \frac{(81x^2-58x+9)^2}{20736x^4(x-1)^4}$$

which is the square of:

$$U^2 - \frac{3x-1}{2x(x-1)} U + \frac{81x^2 - 58x + 9}{144x^2\,(x-1)^2}$$

Since $L^{\circledS 6}$ also has a rational solution $x^2(x-1)^2$, we get from the above proof that $\mathcal{G}(L)$ is $D_3^{SL_2}$ ◇

For $\mathcal{G}(L) \cong D_2^{SL_2}$ one can also use the above approach to compute the special polynomial of minimal degree two. Let J_0 and J_1 be generators of the space of rational solutions of $L^{\circledS 4}$. Either the special polynomial associated with J_1

(or J_0) is a square, or we set $J_\lambda = J_0 + \lambda J_1$ and construct the special polynomial $P_\lambda(u)$ associated with J_λ. Call R_u the resultant in u of $P_\lambda(u)$ and $\frac{\partial}{\partial u}P_\lambda(u)$; then, we must have $R_u = 0$, which gives equations for λ. For $k \subseteq \overline{\mathbb{Q}}(x)$ this resultant R_u is a polynomial in x and λ; the gcd $G(\lambda)$ of the coefficients of R (viewed as a polynomial in x) will have 3 distinct solutions[1] corresponding to the three special polynomials of lowest degree. If one does not look for the minimal polynomial of lowest degree but rather for a solution defined in the same field as the coefficients, one could look for an integer which is not a zero of $G(\lambda)$. Here at most four integers have to be chosen, since there are at most 3 zeros.

Example. - Consider the irreducible equation

$$L(y) = y'' - \left(-\frac{3}{16\,x^2} - \frac{3}{16\,(x-1)^2} + \frac{3}{16\,x\,(x-1)}\right) y = 0$$

The fourth symmetric power has a two dimensional rational solution space generated by $J_0 = x - x^3$ and $J_1 = x^2 - x$. Thus, $\mathcal{G}(L)$ is the quaternion group and we get the following two special polynomials:

$$\begin{aligned} P_{J_0} \;=\;& 256\,(x-1)^4\,x^4\,U^4 - 256\,(2\,x-1)\,(x-1)^3\,x^3\,U^3 \\ &+ 32\,\left(11\,x^2 - 11\,x + 3\right)(x-1)^2\,x^2\,U^2 \\ &- 16\,x\,(x-1)\,(2\,x-1)\left(3\,x^2 - 3\,x + 1\right) U \\ &+ \left(3\,x^2 - 3\,x + 1\right)^2 \end{aligned}$$

and

$$\begin{aligned} P_{J_1} \;=\;& 256\,(x-1)^4\,(x+1)\,x^4\,U^4 - 256\,\left(3\,x^2 - 1\right)(x-1)^3\,x^3\,U^3 \\ &+ 96\,x^2\left(9\,x^3 - 8\,x^2 + 1\right)(x-1)^2\,U^2 \\ &- 16\,\left(27\,x^4 - 45\,x^3 + 22\,x^2 - x - 1\right)(x-1)\,x\,U + 3\,x \\ &- 189\,x^4 + 157\,x^3 - 51\,x^2 + 81\,x^5 + 1 \end{aligned}$$

Then for any constants α and β, the polynomial $\alpha P_{J_0} + \beta P_{J_1}$ is special. Let us now use the methods described above.
First, we check that P_{J_1} is not a square (by computing the gcd of P_{J_1} and its derivative w.r.t U). Then, we set $P_\lambda := P_{J_0} + \lambda P_{J_1}$, we compute R_U, the resultant in U of P_λ and[2] $\frac{\partial}{\partial U}P_\lambda$. There, we have (at least) two possible strategies

Method 1 If we want to find an irreducible special polynomial without increasing the constant field, we search for one of degree 4. Here, P_{J_1} is not a square so

[1] If the solutions are not rational then new constants have to be introduced. The rationality problem is discussed for example in [13, 14]

[2] in factored form, the resultant is $17592186044416\,x^{22}\,(x-1)^{22}\,\lambda^2\,(1+2\,\lambda)^2\,(1+\lambda)^2\,(x\,\lambda + \lambda + 1)$, but we don't need that form for the actual computation

it is irreducible and we are done. If it had been a square, then we would have had to try 3 values for λ that do not annul R_U, one at least would work .

Method 2 We compute the square-free part of the gcd $G(\lambda)$ of the coefficients of R_U; this gives $\lambda + 3\lambda^2 + 2\lambda^3$, which obviously factors into $\lambda(1+\lambda)(1+2\lambda)$. Thus, P_λ is irreducible unless $\lambda \in [0, -1, -1/2]$; performing a square-free decomposition on P_λ, we get that the corresponding polynomials are:

$$\begin{aligned} Q_0 &= U^2 - \frac{(2x-1)U}{2x(x-1)} + \frac{3x^2-3x+1}{16x^2(x-1)^2} \\ Q_{-1} &= U^2 - \frac{(3x-2)U}{2x(x-1)} + \frac{9x^2-11x+3}{16x^2(x-1)^2} \\ Q_{-1/2} &= U^2 - \frac{(3x-1)U}{2(x-1)x} + \frac{9x^2-7x+1}{16x^2(x-1)^2} \end{aligned}$$

In these three polynomials, the reader will easily check that the coefficients of U are the logarithmic derivative of the square root of rational functions $(\sqrt{x(x-1)}, x\sqrt{x-1}, (x-1)\sqrt{x})$ respectively). $\diamond$

4.3 The primitive case

The following shows that for the primitive case it is always possible to look only for rational solutions of symmetric powers. However the algebraic solution of the Riccati found this way will not be of lowest algebraic degree for $A_4^{SL_2}$ and $S_4^{SL_2}$

Lemma 7 *Let $L(y) = 0$ be a second order equation whose differential Galois group G is a finite primitive subgroup of $SL(2, \mathcal{C}_K)$.*

- *If $G \cong A_4^{SL_2}$, then the special polynomial obtained from the logarithmic derivative of the unique rational solution of $L^{\circledS 6}$ is irreducible.*
- *If $G \cong S_4^{SL_2}$, then the special polynomial obtained from the logarithmic derivative of the unique rational solution of $L^{\circledS 8}$ is irreducible.*
- *If $G \cong A_5^{SL_2}$, then the special polynomial obtained from the logarithmic derivative of the unique rational solution of $L^{\circledS 12}$ is irreducible.*

In all cases, it is the special polynomial of lowest order one can construct using rational *solutions of symmetric powers of $L(y)$.*

Proof. - The (abstract) group $A_4^{SL_2}$ has 7 irreducible characters, the trivial one denoted **1**, 2 characters $\zeta_{2,1}$ and $\zeta_{2,2}$ of degree 2 (where the trace of an element of order 3 is different from one and thus the representation is not in

$SL(2, \mathcal{C}_K)$), another character ζ_2 of degree two (corresponding to a representation in $SL(2, \mathcal{C}_K)$) and a character ζ_3 of degree 3. If a second order equation $L(y) = 0$ has Galois group $\mathcal{G}(L) \cong A_4^{SL_2}$, then the corresponding character of $\mathcal{G}(L)$ will be $\chi = \zeta_2$. The character χ_m of $\mathcal{G}(L^{\circledS m})$ can be computed according to the formula given in [10] p. 15:

$$\begin{array}{lll} \chi_2 = \zeta_3 & \chi_4 = \zeta_{1,2} + \zeta_{1,2} + \zeta_3 & \chi_6 = \mathbf{1} + 2\zeta_3 \\ \chi_3 = \zeta_{2,1} + \zeta_{2,2} & \chi_5 = \zeta_{2,1} + \zeta_{2,2} + \zeta_2 & \end{array}$$

Since there are no semi-invariants of degree 2 or 3, the unique special polynomial obtained from the logarithmic derivative of a rational solution of $L^{\circledS 6}$ cannot be the product of special polynomials of lower order.
The proof in the other cases are similar and can be deduced from the decompositions that follow:

- The (abstract) group $S_4^{SL_2}$ has 8 irreducible characters, the trivial one $\mathbf{1}$, another characters $\zeta_{1,1}$ of degree 1, 1 characters ζ_2 of degree 2 which is not faithful, two (conjugated) character $\zeta_{2,0}$ and $\zeta_{2,1}$ of degree 2 (corresponding to representations in $SL(2, \mathcal{C}_K)$), two character $\zeta_{3,1}$ and $\zeta_{3,2}$ of degree 3 and a character ζ_4 of degree 4. For $\zeta_{2,i}$ we set $j \equiv i+1 \pmod 2$ and get:

$$\begin{array}{lll} \chi_2 = \zeta_{3,1} & \chi_5 = \zeta_{2,j} + \zeta_4 & \chi_8 = \mathbf{1} + \zeta_2 + \zeta_{3,1} + \zeta_{3,2} \\ \chi_3 = \zeta_4 & \chi_6 = \zeta_{1,1} + \zeta_{3,1} + \zeta_{3,2} & \\ \chi_4 = \zeta_2 + \zeta_{3,2} & \chi_7 = \zeta_{2,i} + \zeta_{2,j} + \zeta_4 & \end{array}$$

- The (abstract) group $A_5^{SL_2}$ has 9 irreducible characters, the trivial one $\mathbf{1}$, two (conjugated) character $\zeta_{2,0}$ and $\zeta_{2,1}$ of degree 2 (corresponding to two representations in $SL(2, \mathcal{C}_K)$), two character $\zeta_{3,1}$ and $\zeta_{3,2}$ of degree 3, two character $\zeta_{4,1}$ and $\zeta_{4,2}$ of degree 4, a character ζ_5 of degree 5 and a character ζ_6 of degree 6. For $\zeta_{2,i}$ we set $j \equiv i + 1 \pmod 2$ and get:

$$\begin{array}{lll} \chi_2 = \zeta_{3,i} & \chi_6 = \zeta_{3,j} + \zeta_{4,2} & \chi_{10} = \zeta_{3,1} + \zeta_{3,2} + \zeta_5 \\ \chi_3 = \zeta_{4,1} & \chi_7 = \zeta_{2,j} + \zeta_6 & \chi_{11} = \zeta_{2,i} + \zeta_{4,1} + \zeta_6 \\ \chi_4 = \zeta_5 & \chi_8 = \zeta_{4,2} + \zeta_5 & \chi_{12} = \mathbf{1} + \zeta_{3,i} + \zeta_{4,2} + \zeta_5 \\ \chi_5 = \zeta_6 & \chi_9 = \zeta_{4,1} + \zeta_6 & \end{array}$$

Example. - Consider the irreducible equation

$$L(y) = y'' - \left(-\frac{3}{16\, x^2} - \frac{2}{9\,(x-1)^2} + \frac{3}{16\, x\,(x-1)} \right) y$$

This equation is studied in [5] p. 23, where a minimal polynomial of degree 4 of an algebraic solution of the Riccati equation is given. This minimal polynomial corresponds to an exponential solution of $L^{\circledS 4}$ which is not rational, but which

is the square root of a rational function. The same equation is also studied in [9] p. 68 where the minimal polynomial of a solution (not of a logarithmic derivative) is computed.
Using our approach, since $L^{\circledS 4}$ has no rational solution we know that $\mathcal{G}(L)$ is a primitive subgroup of $SL(2, \mathcal{C}_K)$. Since $L^{\circledS 6}$ has a rational solution $x^2(x-1)^2$ we get that $\mathcal{G}(L)$ is the tetrahedral group and that the special polynomial associated with the logarithmic derivative $\frac{4x-2}{x^2-x}$ will be irreducible. This gives the following minimal polynomial for an algebraic solution of the Riccati:

$$U^6 - 2\,\frac{(2\,x-1)}{x\,(x-1)}U^5 + \frac{5\,(64\,x^2-63\,x+15)}{48\,x^2\,(x-1)^2}U^4$$
$$-\frac{5\,(512\,x^3-745\,x^2+351\,x-54)}{432\,x^3\,(x-1)^3}U^3$$
$$+\frac{5\,(4096\,x^4-7840\,x^3+5485\,x^2-1674\,x+189)}{6912\,x^4\,(x-1)^4}U^2$$
$$-\frac{(3645\,x-16254\,x^2+35781\,x^3-38720\,x^4+16384\,x^5-324)}{20736\,x^5\,(x-1)}U$$
$$+\frac{-29889\,x+169209\,x^2-506331\,x^3+842008\,x^4+262144\,x^6-735232\,x^5+2187}{2985984\,x^6\,(x-1)^6}$$

$\diamond$

5 Final remarks

The initial goal of this work was to check for a coefficient field $k \subset \mathbb{C}(x)$ if one really had to perform an extension of the constant field of k to compute a Liouvillian solution to a second order differential equation (cf. [14, 13]). This paper shows that for irreducible second equations no extension of the constant field of k is needed if one not only looks at algebraic solutions of the Riccati of lowest algebraic degree. The paper also shows that the introduction of new constants results in the completely reducible case from the factorization of a special polynomial, which may not be possible over the coefficient field k.

We do not claim that the algorithm presented here is better/faster than the Kovacic algorithm. However we feel that the formulation via rational solutions simplifies the presentation and makes the algorithm easier to implement. Also the algorithm is not limited to the case $k = \mathbb{C}(x)$ and holds for any second order equation with unimodular Galois group (i.e. the special form $y'' - ry$ is not needed).

The result contained in this paper give an overview of all the possible solutions of a second order equation and show how *special* the second order case is by exhibiting some of the properties which make it so *special.*

References

[1] BRONSTEIN M *Linear differential equations : breaking through the order two barrier* Proceedings ISSAC 92, ACM Press 1992.

[2] BRONSTEIN M *Solutions of linear differential equations in their coefficient field* J.Symb.Comp **13**, 1992

[3] DUVAL A & LODAY-RICHAUD M *Kovacic's algorithm and its application to some families of special functions* AAECC Journal **3**, 1992.

[4] KAPLANSKY I *An introduction to differential algebra* Second edition, Hermann, Paris 1976.

[5] KOVACIC J *An algorithm for solving second order linear homogeneous differential equations* J.Symb.Comp (1986) **2** pp 3-43.

[6] SINGER M.F *Liouvillian solutions of linear differential equations* Amer.J.Mat. **103** 1981.

[7] SINGER M.F *Liouvillian solutions of linear differential equations with liouvillian coefficients* J.Symb.Comp (1991) vol **11**, pp 251-273.

[8] SINGER M.F *Testing reducibility of linear differential operators: a group theoretic perspective* Preprint, University of North Carolina, 1994.

[9] SINGER M.F & ULMER F *Liouvillian and algebraic solutions of second and third order linear differential equations* J.Symb.Comp (1993) vol **16**, pp 37-73.

[10] SINGER M.F & ULMER F *Galois groups of second and third order linear differential equations* J.Symb.Comp (1993) vol **16**, pp 1-36.

[11] SINGER M.F & ULMER F *Necessary conditions for liouvillian solutions of (third order) linear differential equations* To appear in AAECC Journal.

[12] SPRINGER T.A. *Invariant Theory* Lecture notes in Math., Springer, 1973.

[13] ULMER F *Linear differential equations of prime degree (the imprimitive case)* Preprint, Université de Rennes, 1994.

[14] VAN DER PUT M & HENDRIKS P *A rationality result for Kovacic's algorithm* Proceedings ISSAC 93, ACM press 1993.

[15] WEIL J.A *The use of the Special semi-groups for solving quasi-linear differential equations* Proceedings ISSAC 94, ACM press 1994.