# First integrals and Darboux polynomials of homogeneous linear differential systems

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**Abstract.** This paper studies rational and Liouvillian first integrals of homogeneous linear differential systems $Y' = AY$ over a differential field $k$. Following [26], our strategy to compute them is to compute the Darboux polynomials associated with the system. We show how to explicitly interpret the coefficients of the Darboux polynomials as functions on the solutions of the system; this provides a correspondence between Darboux polynomials and semi-invariants of the differential Galois groups, which in turn gives indications regarding the possible degrees for Darboux polynomials (particularly in the completely reducible cases). The algorithm is implemented and we give some examples of computations.

## 1 Introduction

Consider the equation $L(y) = y'' + y = 0$. If $y$ is any solution of $L(y) = 0$, it is easy to verify that the total derivative of $(y')^2 + y^2$ is zero. Thus, for any solution $y$ of $L(y) = 0$, there will exist some constant $c$ such that $(y')^2 + y^2 = c$. In that case, we say that $(y')^2 + y^2$ is a *first integral* of $L$ (see below for some more precise definitions). The aim of this paper is to study a procedure allowing one to decide whether or not a given system of linear differential equations admits a first integral.

Our strategy to compute first integrals will be the use of the notion of Darboux polynomials (see definition below). In [26], we gave a partial procedure for computing Darboux polynomials of given degree for linear differential equations; as systems can be converted to equations, this theoretically included the case of systems. However, this conversion produces intermediate equations with "huge" coefficients, so in this paper we show how to handle systems directly (without converting to equations); we also study how to make good use of some degenerate situations, which improves our first algorithm from the practical point of view.

In general, finding the degree of Darboux polynomials is a very difficult problem (see e.g [12]). The main result of this paper is the ability to characterize the

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Darboux polynomials and their degree by relating them bijectively with the semi-invariants of the differential Galois group. Thus, all results from invariant/representation theory are at our disposal and this provides bounds on the degrees of the Darboux polynomials in many cases.

The paper is organized as follows: in the rest of this section, we recall some of the properties of the Darboux polynomials of linear differential systems; in section 2, we give a characterization of the coefficients of the Darboux polynomials (proposition 7); in section 3, we show how this characterization provides a correspondence between the Darboux polynomials and the semi-invariants of the differential Galois group (theorem 12). In section 4, we use this material to design computational procedures and, in section 5, we conclude with some examples and remarks.

### 1.1 Rational first integrals

Let $(k, \partial_k)$ be a differential field with an algebraically closed[2] constant field $\mathcal{C}$; we will often denote the derivation by the usual symbols ', '', etc. We assume that $k$ has the following property: given a linear differential equation $L$ with coefficients in $k$, we must have an algorithm that finds the rational and the exponential solutions of $L$ over $k$ (recall that a solution is called exponential over $k$ if its logarithmic derivative lies in $k$). An example of such a field is $C(x)$, where $C$ is any number field of characteristic 0, with the usual derivation $\frac{d}{dx}$ (see [4], and [16] for a wider class of fields).

In this paper, we consider the following first order linear differential system:

$$(A): \qquad Y' = AY \quad \text{with} \quad A \in \mathcal{M}_{n,n}(k) \tag{1}$$

Let us first introduce the notion of rational first integral. One can algebraically modelise a "generic solution" of the system $(A)$ the following way. Consider some indeterminates $(y_1, \ldots, y_n)$ and form the field $k(y_1, \ldots, y_n)$. Let $A_i$ denote the $i$-th row of $A$ and let $Y = (y_1, \cdots, y_n)^t$. Then, one can form the derivation

$$D_k = \partial_k + A_1 Y \frac{\partial}{\partial y_1} + A_2 Y \frac{\partial}{\partial y_2} + \ldots + A_n Y \frac{\partial}{\partial y_n} \tag{2}$$

where $\partial_k$ denotes the derivation of the coefficients of an element of $k(y_1, \ldots, y_n)$. This derivation turns $k(y_1, \ldots, y_n)$ into a differential field of rational functions on solutions of $(A)$. Thus, we will say that an element $M \in k(y_1, \ldots, y_n)$ (with $M \notin \mathcal{C}$) is a *rational first integral* for $(A)$ if $D_k M = 0$ (i.e, for any solution $Y$ of $(A)$, $M(Y)$ is a constant). When no confusion is possible, we will simply write $D$ instead of $D_k$.

[2] This assumption will usually not be used for practical computations, but it is absolutely necessary to prove theorems using differential Galois theory.

### 1.2 The Darboux polynomials of $(A)$

Consider the ring $k[y_1, \ldots, y_n]$; this is also turned into a differential ring by $D$. Suppose that $M \in k(y_1, \ldots, y_n)$ is a rational first integral of $(A)$. Then, $M$ can be written as a quotient $M = \frac{F}{G}$ with $F, G \in k[y_1, \ldots, y_n]$ and $F, G$ relatively prime. Then, $DM = 0$ implies that $D(F)G - D(G)F = 0$. This in turn implies that $F$ divides $DF$. In other terms, there exist $\alpha \in k[y_1, \ldots, y_n]$ such that $DF = \alpha F$.

**Definition 1.** Let $F \in k[y_1, \ldots, y_n]$. We say that $F$ is a *Darboux polynomial* for $(A)$ if there exists $\alpha \in k[y_1, \ldots, y_n]$ such that $DF = \alpha F$.

Relating the search for first integrals to the computation of Darboux polynomials is an old method[3]. We will now recall some of the essential properties of Darboux polynomials (see [26, 27] for more details and references). Let $\mathcal{S}_{A,k}$ denote the set of the Darboux polynomials for $(A)$ with coefficients in $k$ (denoted simply $\mathcal{S}$ when no confusion is possible). Then, any element of $k$ is obviously in $\mathcal{S}$. Also note that, if $k \subset K$ and $M \in k[y_1, \ldots, y_n]$ is Darboux for $D_K$, then $M$ is Darboux for $D_k$ (see e.g [24])

**Lemma 2.** *[12, 26, 24] The set $\mathcal{S}$ of Darboux polynomials is a semi-group: if $F, G \in \mathcal{S}$, then $FG \in \mathcal{S}$. Moreover, if $F \in \mathcal{S}$ then all its irreducible factors are in $\mathcal{S}$.*

Thus, to describe $\mathcal{S}$, it will be enough to compute its irreducible elements. In the case of the derivation $D$ defined above, we have additional information because $D$ is a homogeneous and degree 0 application: if one picks a monomial of degree $m$ in the $y_i$, then its derivative (by D) is easily seen to be a homogeneous polynomial of the same degree $m$. As a consequence (see [26]):

**Lemma 3.** *[26] If $M$ is a Darboux polynomial for $(A)$ verifying $DM = \alpha M$, then $\alpha \in k$ and every homogeneous component $M_i$ of $M$ (taken as a multivariate polynomial in the $y_i$) also verifies $DM_i = \alpha M_i$.*

In the sequel, this lemma will allow us to consider only homogeneous irreducible Darboux polynomials. Before we introduce our next structure lemma, we need a definition:

**Definition 4.** A differential extension $K \supset k$ is called a *Liouvillian* extension if there exists a tower of extensions $k = k_0 \subset k_1 = k(\theta_1) \subset \ldots \subset k_n = k(\theta_1, \cdots, \theta_n) = K$ such that we have $\theta_i$ algebraic over $k_{i-1}$ or $\theta_i' \in k_{i-1}$ or $(\theta_i')/\theta_i \in k_{i-1}$ (for all $i = 1, \cdots, n$).
An element is said to be Liouvillian over $k$ if it belongs to a Liouvillian extension of $k$.
We will say that $(A)$ has a *liouvillian first integral* if it has a polynomial first integral over a liouvillian extension of $k$.

[3] They appear with many names in the literature. Depending on the papers, they also appear as "special polynomials", "invariant algebraic curves", "eigenpolynomials", or "algebraic solutions".

We refer the reader to [15, 14, 20, 24, 6, 2] for more properties of Liouvillian extensions; see also [17], where a more general definition of Liouvillian first integrals is studied. In the sequel, we will use the following structure result (see [27] for a proof and more details):

**Lemma 5.** *[27] Let $K \supset k$ be a Liouvillian extension of $k$. Then, the derivation $D_K$ admits a Darboux polynomial over $K$ if and only if $D_k$ admits a Darboux polynomial over $k$.*

*Remark.* Note that, if $DM = \alpha M$ and $f' = -\alpha f$, then $D(fM) = 0$. If $f \in k$, then this means that $fM$ is a polynomial first integral of $(A)$; else, this means that $\alpha = e^{\int f}$ and $fM$ is a Liouvillian first integral of $(A)$. In fact lemma 5 shows that, for us, these will be the only interesting type of Liouvillian first integrals.

## 2 Duality and Darboux polynomials

In this section, we will show how to characterize the Darboux polynomials in terms of constructions on the matrix $A$. This will enable us to interpret the coefficients of Darboux polynomials in terms of solutions of $(A)$.

**Solutions.** Let us first focus on the notion of solutions. In the case of a linear differential system, there is a notion of minimal differential extension containing a fundamental set of solutions of $(A)$:

**Definition 6.** A differential field extension $K \supset k$ is said to be a *Picard-Vessiot extension* for $(A)$ if

1. $K = k(Y_{1,1}, \ldots, Y_{1,n}, \ldots, Y_{n,n})$, where $Y_1, \ldots, Y_n$ is a fundamental set of solutions of $(A)$ (i.e $K$ is the differential field obtained by adjoining to $k$ the components of the vectors $Y_1, \ldots, Y_n$).
2. $K$ and $k$ have the same field of constants.

As the constant field of $k$ is algebraically closed of characteristic 0, one can show that Picard-Vessiot extensions exist and are unique up to differential isomorphism (cf. [6] p.21 and [7]). In the sequel, the term "solution" will always denote a solution in the Picard-Vessiot extension $K$.

Consider the $n$-dimensional $\mathcal{C}$-vector space $V$ of solutions of $(A)$ and denote by $U$ a *fundamental solution matrix* (i.e the columns $w$ of $U$ are solutions of $w' = Aw$ and they span $V$). We will call *construction on $V$* a vector space obtained from $V$ by successive use of the following operations: taking the dual, tensor products, direct sums, symmetric and exterior powers. To any construction $Const(V)$, there is a corresponding linear differential system having $Const(V)$ as its solution space (see e.g [2, 11]). In this paper, we only use the dual and the symmetric powers, which we will now make explicit for the reader's convenience.

**The dual system** We construct a system whose solution space is isomorphic to $V^*$, the dual space of $V$. Indeed, consider the matrix $(U^{-1})^t$; it is well defined because the columns of $U$ span $V$ and thus $\det(U)$ does not vanish. Then, for any column $w_i$ of $U$ and any column $v_i$ of $(U^{-1})^t$, we have by construction $\langle v_i, w_j \rangle = \delta_{i,j}$ (the Kronecker symbol); it follows that the columns of $(U^{-1})^t$ span $V^*$. Now, using the relation $U.U^{-1} = I$, one easily finds that $(U^{-1})^t$ satisfies $Y' = -A^t Y$; in the sequel, we will use the notation $A^* = -A^t$ to denote the matrix of a dual system.

**Symmetric powers** Let $Y = (y_1, \cdots, y_n)^t$ denote a solution of $Y' = AY$. If we consider a monomial $M$ of degree $m$ in the $y_i$, then $DM$ is a linear combination of monomials of degree $m$. As there are $\nu = \binom{n+m-1}{n-1}$ possible monomials of degree $m$ in $n$ variables, we obtain that the vectors $w = (y_1^m, \cdots, y_{n-1} y_n^{m-1}, y_n^m)$ of all monomials of degree $m$ in the $y_i$ satisfy a $\nu \times \nu$ system which we denote by $Y' = S^m(A)Y$. The reason for this notation is that one can show (cf [3]) that its $\nu$-dimensional solution space is isomorphic to $S^m(V)$, the $m$-th symmetric power of $V$ (see e.g [9] p. 635 for definition and properties of symmetric powers).

**Back to Darboux polynomials.** These constructions enable the following characterization of Darboux polynomials:

**Proposition 7.** *The system $(A)$ admits a Darboux polynomial of degree $m$ with a vector $v$ of coefficients if and only if there exists a non-zero $f$ exponential over $k$ (i.e $f'/f \in k$) such that $fv$ is a solution of the system $Y' = (S^m(A)^*)Y$.*

*Proof.* Let $M = v_\nu y_n^m + v_{\nu-1} y_n^{m-1} y_{n-1} + \ldots + v_1 y_1^m$ be a polynomial first integral of $(A)$ and let $v$ denote its coefficient vector (i.e $v = (v_1, \cdots, v_\nu)$). Then, for any solution $Y$ of $(A)$, we have $M(Y) \in \mathcal{C}$. But now, if we let $w = (y_1^m, \cdots, y_{n-1} y_n^{m-1}, y_n^m)$, then we have $M(Y) = \langle v, w \rangle$; thus, we obtain $\langle v, w \rangle \in C$ and thus $v \in S^m(V)^*$. It follows that $v$ is a solution of $(S^m(A)^*)$ if and only if $M$ is a polynomial first integral.
Now, let $M$ be a homogeneous Darboux polynomial of degree $M$ with $DM = \alpha M$ and let $v$ denote the vector of its coefficients. Consider a minimal extension $k_1 = k(f)$ of $k$ containing an element $f$ such that $f' = -\alpha f$ (i.e if $k$ contains such an element, then $k_1 = k$). Then, it is immediately seen that $D(fM) = 0$. Thus, the above construction shows that $fv$ is a solution of the system $Y' = (S^m(A)^*)Y$. Conversely, suppose that $fv$ is a solution of the system $Y' = (S^m(A)^*)Y$ with $f$ exponential over $k$ ($f \neq 0$) and $v \in k^\nu$; let $M$ denote as above the polynomial whose coefficient vector is $v$. The relation $D(fM) = 0$ is then clearly equivalent to $DM = -\frac{f'}{f} M$ and we are done. □

*Remark.* Using the language of vector spaces with a connection, some ideas in this direction were also suggested in [10].

This characterization will be the key point for the algorithm of section 4. In the following section, we will show how it may also provide a characterization of the degrees of the Darboux polynomials.

## 3 Darboux polynomials and semi-invariants of the differential Galois group

In this section, we recall and state some useful notions about differential Galois theory (see e.g [6, 15, 11, 2]). In particular, we will show the link between first integrals and semi-invariants of the differential Galois group; only the new results are proved.

### 3.1 Differential Galois theory

Many of the properties of the solutions of $(A)$ derive from the fact that there is a group action on the vector space of solutions that induces a "differential Galois theory". To follow the frame of classical Galois theory, the Picard-Vessiot extensions will play the role of a splitting field for $(A)$.

**Definition 8.** The differential Galois group of a differential extension $K \supset k$ is defined as the group of automorphisms of $K$ that commute with the derivation and that leave $k$ pointwise fixed.
The *differential Galois group* $G$ of $(A)$ is defined as the differential Galois group of $K/k$, where $K$ is a Picard-Vessiot extension of $k$ for $(A)$.

The main property that we will use is the following standard lemma (see e.g [15, 19]):

**Lemma 9.** *If $y \in K$, then $y \in k$ if and only if $g(y) = y$ for all $g \in G$. If $y \neq 0$, then $y'/y \in k$ if and only if for all $g \in G$ there exists a constant $\psi_y(g) \in C$ such that $g(y) = \psi_y(g)y$.*

If we choose a fundamental set of solutions $\{Y_1, Y_2, \ldots, Y_n\}$ of the system $(A)$, then for each $\sigma \in G$ we get $\sigma(Y_i) = \sum_{j=1}^{n} c_{ij} Y_j$ , where $c_{ij} \in \mathcal{C}$. This gives a faithful representation of $G$ as a subgroup of $GL(n, \mathcal{C})$ (in fact, $G$ is a linear algebraic subgroup of $GL(n, \mathcal{C})$). Different choices of bases give equivalent representations. In the sequel, we always consider this equivalence class of representation as *the* representation (module) of $G$.

Let $V$ be again the solution space of $(A)$; the action of $G$ induces a structure of $G$-module on $V$. One can show that any construction on $V$ is also a $G$-module (see [11] p.134). For example, $G$ acts naturally on the dual $V^*$ and on $V \otimes V$ in the following way. Let $g \in G$ and $\sigma$ be its representation on $V$; the action of $G$ on $V^*$ is defined by $\langle g(u), y \rangle = \langle u, g^{-1}(y) \rangle$ for $u \in V^*, y \in V$ (see e.g [2, 11]). The representations of $g$ on $V^*$ and on $V \otimes V$ are respectively $\left(\sigma^{-1}\right)^t$ and $\sigma \otimes \sigma$. The reader may consult [15, 6, 8, 11] for proofs and more properties of the Galois group.

Computing the Galois group is, in general, an open problem. Algorithms exist for n=2 (cf [24, 19] and references therein) and n=3 (cf [19]); see also [11] for a survey of other important methods.

**Systems of the same type.** Consider the system $Y' = AY$ and suppose we perform a change of variables $Z = PY$ with $P \in GL(n, k)$. Then,

$$Z' = PY' + P'Y = (PAP^{-1} + P'P^{-1})Z. \tag{3}$$

We will say that two systems are *of the same type* (or *equivalent*) if such a relation (with $P$ invertible) holds between them.

Note that if $U$ is a fundamental matrix for $(A)$, then $PU$ is a fundamental matrix for $(PAP^{-1} + P'P^{-1})$ and the entries of $PU$ are in the same Picard-Vessiot extension as those of $U$. Looking at the way the Galois group acts on the solutions, we get a standard property that will be crucial in the sequel: two systems of the same type have the same Galois group (see, e.g [11] or lemmas 2.5 and 2.6 in [18]).

In the sequel, the operation of changing to a system of the same type will be called a *$G$-change of variables*. A property that we will use without further mention is the following intuitive lemma:

**Lemma 10.** *Let $\tilde{\phi} : y \mapsto \tilde{y}$ be a $G$-change of variable. Then, the system $(A)$ defined by (1) has a Darboux polynomial $M$ if and only if $\tilde{\phi}(M)$ is a Darboux polynomial for $(\tilde{A})$.*

*Proof.* If $V$ and $\tilde{V}$ are isomorphic $G$-modules, then $S^m(V^*)$ and $S^m(\tilde{V}^*)$ are also isomorphic $G$-modules. Thus, the systems $S^m(A)^*$ and $S^m(\tilde{A})^*$ are equivalent and one has a solution $fv$ with $f'/f \in k$ if and only if the other one has the solution $f\tilde{v}$. □

**Equations and systems.** For algorithmic issues, we sometimes need to convert systems to equations and vice-versa. We now briefly review the standard ways of performing this task (see also [11, 2]). Consider an ordinary homogeneous linear differential equation

$$L(y) = y^{(n)} - a_{n-1}y^{(n-1)} - \cdots - a_1y' - a_0y = 0 \qquad (a_i \in k). \tag{4}$$

Then, solving this equation is equivalent to solving the companion system

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_{n-1} \\ y_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \cdots \\ y_{n-1} \\ y_n \end{pmatrix}$$

Conversely, suppose that $(y_1, y_2, \cdots, , y_n)$ satisfy a homogeneous first order linear differential system $(A)$ of size $n$. To find an equation associated with $(A)$, we would like to find a system of the same type (in the above sense) in companion form; this is done by the following cyclic vector process (see e.g [1, 11, 2] for references and other methods). Consider $\Lambda \in k^n$ and let $z_1 = \Lambda Y =$

$\lambda_1 y_1 + \ldots + \lambda_n y_n$. We compute $z_2 = z_1', \ldots, z_{n+1} = z_1^{(n)}$ by using the relation $Y' = AY$. We obtain $n+1$ linear expressions in the $n$ variables $y_i$ and so they are linearly dependant: this provides a linear differential equation $\mathcal{L}(z_1) = 0$ for $z_1$. Letting $Z = (z_1, \cdots, z_n)^t$, we now have a relation $Z = PY$ and $Z' = BY$; If the matrix $P$ is invertible, $\Lambda$ is called a *cyclic vector* for the system and $Z' = (BP^{-1})Z$ is a companion system of the same type as $(A)$. It can be shown that the cyclic vectors form a Zariski open set, and almost all choices of $\Lambda$ will fit.

Thus, in the following, everything that is stated for first order systems is valid for n-th order equations and vice-versa (for example, we call Darboux polynomial of an equation a Darboux Polynomial of the associated companion system).

### 3.2 Invariants and semi-invariants of the differential Galois group

We now show how to characterize Darboux polynomials in terms of representation of the Galois group. Let $V$ be a $\mathcal{C}$ vector space with basis $y_1, \ldots, y_n$ and $G \subseteq GL(V)$ a linear group. One defines an action of $G$ on the symmetric algebra $S(V)$ of $V$ ($S(V) \approx k[Y_1, \ldots, Y_n]$) by $g \cdot (p(Y_1, \ldots, Y_n)) = p\,(g(Y_1), \ldots, g(Y_n))$.

**Definition 11.** A polynomial $P$ with the property that

$$\forall g \in G, \quad g\,(P(Y_1, \ldots, Y_n)) = \psi_P(g) \cdot (P(Y_1, \ldots, y_n))\,, \text{ with } \psi_P(g) \in \mathcal{C} \quad (5)$$

is called a *semi-invariant* of $G$ of degree $\deg(P)$ (where $\deg(P)$ is the total degree). If $\forall g \in G$ we have $\psi_P(g) = 1$, then $P(Y_1, \ldots, Y_n)$ is called an *invariant* of $G$.

*Remark.* This definition of invariants is the classic one. However, several authors also call invariants elements of *any* construction on $V$ that are left invariant by $G$.

As the action of $G$ is homogeneous, the invariants (resp. semi-invariants) are generated by homogeneous invariants (resp. semi-invariants). Therefore, they will be found in the symmetric powers $S^m(V)$ of $V$. From the definition, we get the following nice characterization of Darboux polynomials:

**Theorem 12.** *Let $G_*$ denote be the differential Galois group of the system $Y' = A^*Y$. Up to scalar multiplication, there is a one to one correspondence between the Darboux polynomials (resp. polynomial first integrals) of $(A)$ and the semi-invariants (resp. invariants) of $G_*$.*

*Proof.* Let $M$ be a Darboux polynomial of degree $m$ and denote by $Sol(S^m(A^*))$ the solution space of $(S^m(A^*))$; by construction, we have a $G_*$-isomorphism $\phi_m : S^m(V^*) \mapsto Sol(S^m(A^*))$. By proposition 7, the vector $v$ of coefficients of $M$ is such that there exists $f$ with $f'/f \in k$ and $f.v \in Sol(S^m(A^*))$; thus, if we call $z_1, \cdots, z_n$ a basis of $V^*$, there exists some homogeneous $P(z_1, \cdots, z_\nu) \in S^m(V^*)$ such that $\phi_m(P) = f.v$. By lemma 9, $v$ is left fixed by $G_*$ and, for all $g \in G_*$,

there exists a constant $\psi(g)$ such that $g(f) = \psi(g)f$. As $\phi_m$ commutes with any $g \in G_*$, we deduce that $g(P) = \psi(g)P$ for all $g \in G_*$ and so $P$ is a semi-invariant. Conversely, let $P$ be a semi-invariant and put $u = \phi_m(P)$ (i.e $u = (u_1, \cdots, u_\nu) \in Sol(S^m(A^*))$). Select $i$ such that $u_i \neq 0$. For all $g \in G$, we have $g(u) = \psi(g)u$ so that $g(u_i) = \psi(g)u_i$. If we put $f = u_i$, lemma 9 show that $f'/f \in k$. Now, for all $j = 1, \cdots, \nu$, we have $g(\frac{u_j}{f}) = \frac{\psi(g)}{\psi(g)}\frac{u_j}{f}$. Therefore, if we let $v = (u_1/f, \cdots, u_\nu/f)$, we have $v \in k^\nu$ such that $f.v \in Sol(S^m(A^*))$ and proposition 7 shows that $v$ is the vector of coefficients of a Darboux polynomial. □

This result allows us to use representation theory to find bounds on the degree of Darboux polynomials. For systems that have a Liouvillian solution, bounds have been given by Singer ([14]) and improved/generalized by Ulmer ([23, 20]). Using representation theory, Singer and Ulmer have obtained sharp bounds for $n = 2$ and $n = 3$ (for $n = 2$ older bounds existed, see the references in [20, 24]). Other comments on bounds can be found in [27]. For systems of size $n > 3$, we don't know yet how to bound the degree of a semi-invariant of minimal degree in all cases: this is still an interesting (difficult) open problem.

### 3.3 Invariants of completely reducible systems.

In this part, we show that, with an assumption on the Galois group, one can restate the above proposition directly in terms of $G$. Let $V$ be a C-vector space and $G \in GL(V)$. We denote by $\mathrm{Inv}_V(G)$ the G-subspace of elements of $V$ that are left invariant by all elements of $G$. We will say that $G$ is *reductive* (or *completely reducible*) if all constructions on $V$ are completely reducible $G$-modules (i.e any $G$-invariant subspace has a complementary $G$-invariant subspace); we say that a system is completely reducible if it has a reductive Galois group.

**Proposition 13.** *Consider a linear differential system with a reductive Galois group $G$ and solution space $V$; let $G_*$ denote the Galois group of the dual system whose solution space is $V^*$. Then, $G$ has $s$ invariants (resp. semi-invariants) of degree $m$ if and only if $G_*$ has $s$ invariants (resp. semi-invariants) of degree $m$.*

In order to prove this, we will need the following three lemmas:

**Lemma 14.** *The invariants of $G$ on $V^*$ satisfy* $\mathrm{Inv}_{V^*}(G) = \mathrm{Hom}_G(V, \mathcal{C})$

*Proof.* We have $u \in \mathrm{Hom}_G(V, \mathcal{C})$ if and only if $\langle u, g(y)\rangle = g(\langle u, y\rangle)$ for all $y \in V$ and all $g \in G$. As $\langle u, y\rangle \in \mathcal{C}$, we have $g(\langle u, y\rangle) = \langle u, y\rangle$. Now, we have $\langle u, g(y)\rangle = \langle g^{-1}(u), y\rangle$, and thus: $\langle u, g(y)\rangle = g(\langle u, y\rangle)$ if and only if $u = g^{-1}(u)$ for all $g \in G$, which is true if and only if $u$ is an invariant of $G$ in $V^*$. □

**Lemma 15.** *Suppose that $G$ is reductive. Then,* $\mathrm{Inv}_{V^*}(G)$ *is $G$-isomorphic with* $\mathrm{Inv}_V(G)$.

*Proof.* Let $V_1 = \mathrm{Inv}_V(G)$. As $G$ is reductive, there exist a $G$-submodule $V_2$ such that $V = V_1 \oplus V_2$. We embed $V_1^*$ into $V^*$ by letting $u_1 \in V_1^*$ send $V_2$ to 0, so that $V^* = V_1^* \oplus V_2^*$. Consider $u \in V_1^*$. Any $y \in V$ can be expressed as $y = y_1 + y_2$ with $y_i \in V_i$; thus, $g(\langle u, y\rangle) = \langle u, y\rangle = \langle u, y_1\rangle$. Now, we have $\langle u, g(y)\rangle = \langle u, g(y_1) + g(y_2)\rangle = \langle u, g(y_1)\rangle$ (because the $V_i$ are $G$-invariant). But, as $y_1$ is an invariant of $G$, this implies that $\langle u, g(y)\rangle = \langle u, y_1\rangle = g(\langle u, y\rangle)$. By lemma 14, this implies $u \in \mathrm{Inv}_{V^*}(G)$. If $V_1^* \subset_{\neq} \mathrm{Inv}_{V^*}(G)$ then, as $V_1^{**} = V_1$, the dimensions would be such that $V_1 \subset_{\neq} \mathrm{Inv}_V(G)$, a contradiction. Thus, $V_1^* = \mathrm{Inv}_{V^*}(G)$.
We may verify that $V_1$ and $V_1^*$ are $G$-isomorphic; let $f : V_1 \to V_1^*, y_i \mapsto u_i$ (where $y_i, u_i$ denote basis elements) be the standard isomorphism. As $y_i$ and $f(y_i)$ are invariants of $G$, $f(g(y_i)) = f(y_i) = g(f(y_i))$, and $f$ is the desired $G$-isomorphism. □

**Lemma 16.** *The $G$-modules $(S^m(V))^*$ and $S^m(V^*)$ are $G$-isomorphic.*

*Proof.* There is a classical (functorial) isomorphism between $V^* \otimes V^*$ and $(V \otimes V)^*$ (see [9] p. 567). Now, if $\sigma$ is a matrix, it is checked the same way that $\left((\sigma \otimes \sigma)^{-1}\right)^t = \left(\sigma^{-1}\right)^t \otimes \left(\sigma^{-1}\right)^t$; thus, $G$ has the same representation on $V^* \otimes V^*$ and $(V \otimes V)^*$, and so they are $G$-isomorphic. Then, an immediate induction yields the result. □

*Proof of proposition 13.* Suppose that $G$ has $s$ semi-invariants of degree $m$. This means that there are $s$ 1-dimensional $G$-invariants submodules $V_i$ and a $G$-submodule $W$ of $S^m(V)$ such that $W$ has no one-dimensional submodule and $S^m(V) = V_1 \oplus \ldots \oplus V_s \oplus W$. Thus (as in the proof of lemma 15), we have $S^m(V)^* = V_1^* \oplus \ldots \oplus V_s^* \oplus W^*$; by lemma 16, $S^m(V)^*$ is $G$-isomorphic with $S^m(V^*)$; as the $V_i^*$ are 1-dimensional $G$-module, lemma 9 shows that the generators of the $V_i^*$ are exponential over $k$; thus, these are also 1-dimensional $G_*$-modules.
Suppose that $G$ has $s$ invariants of degree $m$, i.e $dim_{\mathcal{C}}\left(\mathrm{Inv}_{S^m(V)}(G)\right) = s$. By lemma 15, we have a $G$-isomorphism between $\mathrm{Inv}_{S^m(V)}(G)$ and $\mathrm{Inv}_{(S^m(V))^*}(G)$. By lemma 16, the latter is $G$-isomorphic with $\mathrm{Inv}_{S^m(V^*)}(G)$ and we obtain that $dim_{\mathcal{C}}\left(\mathrm{Inv}_{S^m(V^*)}(G)\right) = s$. □

Remark: if $G$ is not reductive, then the result is no longer true. For example, consider the operator $L = (\partial^2 - x)(x\partial - 1)$ i.e $L(y) = xy''' - 3y'' - x^2y' + xy$. Then, the equation $L(y) = 0$ has the solution $y = x$, but it can be checked that $L^*(y) = 0$ has no rational solution (where $L^*$ denotes the *adjoint* equation whose solution space is the dual of the solution space of $L$, cf [13]).

**Corollary 17.** *Assume that the system $(A)$ has a reductive Galois group $G$. Up to scalar multiplication, there is a one to one correspondence between the Darboux polynomials (resp. polynomial first integrals) of $(A)$ and the semi-invariants (resp. invariants) of $G$.*

*Proof.* This follows from theorem 12 and proposition 13. □

# 4 Algorithmic issues

## 4.1 The algorithm

Let $M = v_\nu y_n^m + v_{\nu-1} y_n^{m-1} y_{n-1} + \ldots + v_1 y_1^m$ be a Darboux polynomial of given degree $m$ and denote as usual by $v = (v_1, \cdots, v_\nu)^t$ the vector of its coefficients. Assuming that $m$ is given (by the bounds of the previous section), proposition 7 provides the following algorithm for computing $v$.

1. Compute the matrix $\mathcal{A} = S^m(A^*)$.
2. Take a cyclic vector and compute the $G$-change of variables $P$ that makes the system equivalent to a companion form, i.e to a homogeneous linear differential equation $\mathcal{L}_m$.
3. Compute the solutions of $\mathcal{L}_m$ whose logarithmic derivative is in $k$. For any such solution $f$, go to next step; else, return(0).
4. Apply $P^{-1}$ to derive the corresponding vector $V = fv$ with $v \in k^\nu$. Then, $v$ is the coefficient vector of a Darboux polynomial.

In [26] (section 5.1), we gave a method for computing Darboux polynomials of linear differential equations (and this method can be adapted to systems); it is in fact a subclass of the above algorithm, as it corresponded to always choose $(0, \cdots, 0, 1)$ as a candidate cyclic vector. This yielded degeneracies that were difficult to deal with; this problem does not occur with the above algorithm (and we will see below how to take advantage of the degeneracies).

*Remark.* Cyclic vectors are not the only way to solve linear differential systems; for example, in [1], Barkatou gives an algorithm for computing a "rational normal form" that decouples the system into independent linear differential equations of lower order. This form can be used as an alternative to step (2) of our algorithm.

## 4.2 Degenerate cases

If we take a putative cyclic vector $\Lambda$ , then the corresponding $G$-change of variables $P$ would be a matrix with rows $P_i$ satisfying $P_1 = \Lambda$ and $P_{i+1} = P_i' + P_i\mathcal{A}$. If the matrix $P$ is invertible, then the solution spaces of $(\mathcal{A})$ and $\mathcal{L}_m$ are isomorphic. However, if $P$ does not have full rank, $\mathcal{L}_m$ has order less than $\nu$ and we theoretically cannot deduce the solutions of $(\mathcal{A})$ from those of $\mathcal{L}_m$. In this subsection, we show how one can in fact take advantage of this situation and how our algorithm can be improved in this case.

Let $\ker(P)$ have dimension $r > 0$ and compute a basis $V_1, \cdots, V_r$ of $\ker(P)$. For all $i, j$ we have $P_i V_j = 0$. Now, on one hand, we have $P_{i+1} V_j = P_i' V_j + \mathcal{A} V_j = 0$; on the other hand, we have $(P_i V_j)' = 0 = P_i' V_j + P_i V_j'$; thus, we obtain that $P_i(V_j' - \mathcal{A} V_j) =$ and so, for all j, there are elements $c_{i,j} \in k$ such that $V_j' - \mathcal{A} V_j = \sum_{i=1}^r c_{i,j} V_i$. We compute these elements.

First, let us assume for simplicity that $\mathcal{L}_m$ has no exponential solution. If $V$ is an exponential solution of $(\mathcal{A})$ then, by construction, $V$ must satisfy $PV = 0$;

Writing $V = \sum \gamma_i V_i$ in the basis $(V_i)$ of $\ker(P)$, we obtain that $V' - \mathcal{A}V = 0$ if and only if

$$\sum_{i=1}^{r} \left( \gamma_i' + \sum_{j=1}^{r} \gamma_j c_{i,j} \right) V_i = 0.$$

As the $V_i$ form a basis of $\ker(P)$, this yields a system $\Gamma' = C\Gamma$ (with $C = (-c_{i,j})_{i,j}$) for the $\gamma_i$. Thus, we have reduced our $\nu$-dimensional problem to finding the exponential solutions of the $r$-dimensional system $\Gamma' = C\Gamma$.

*Remark.* If $\dim(\ker(P)) = 1$ then we directly have that $V_1$ is the coefficient vector of a Darboux polynomial. Also, note that if $\Gamma' = C\Gamma$ has an exponential solution, then this yields a Darboux polynomial even if $\mathcal{L}_m$ has exponential solutions. Therefore, the above construction leads to a notable improvement of the algorithm (which should be performed before even solving $\mathcal{L}_m$).

Assume that $\mathcal{L}_m$ has an exponential solution $f$ and let $F = (f, \cdots, f^{(\nu-1)})^t$. Letting $V_0$ be a particular solution of $PV = F$, we obtain a general solution (of $PV = F$) in the form $V = V_0 + \sum \gamma_i V_i$. As $P_{i+1}V = f^{(i)} = (P_i V)'$, we obtain again that $P_i'V + P_i\mathcal{A}V = P_i'V + P_iV'$ and thus $V' - \mathcal{A}V \in \ker P$; in particular, we find constants $s_i$ such that $V_0' - \mathcal{A}V_0 = \sum_{i=1}^{r} s_i V_i$ and derive an $r \times r$ inhomogeneous system $\Gamma' = C\Gamma + S$ for the $\gamma_i$. Such a system can be also solved by a cyclic vector process. Note that the produced equation will then be inhomogeneous with an exponential right-hand side and the techniques from [16] must then be used.

## 5 Examples and remarks

The algorithm has been implemented in the computer algebra system MAPLE. Let us see some examples of computations.

### 5.1 Some easy examples

Let $L$ be a linear differential equation and $(A)$ the corresponding system. The *m-th symmetric power* of $L$, noted $L^{\circledS m}$, is the equation produced from $S^m(A)$ by using the (putative) cyclic vector $(1, 0, \cdots, 0)$ (cf [14, 19, 20]). This equation has its solution space spanned by all monomials of degree $m$ in the solutions of $L$.

Consider the equation $L(y) = y'' + y = 0$. Then, its symmetric square is $L^{\circledS 2}(y) = y''' + 4y'$. This has the solution $y = 1$, and thus the Galois group has an invariant of degree 2. The corresponding first integral is $(y')^2 + y^2$.

More generally[4] consider the equation $l(z) = z'' - rz$ with $r \in k$ and let $z_1, z_2$ be a basis for the solution. We can form the equation $L(y) = l^{\circledS 2}(y) = y''' - 4ry' - 2r'y$ with $(y_1 = z_1^2, y_2 = z_1 z_2, y_3 = z_2^2)$ as a basis for its solution

[4] I am indebted to M.F Singer for having suggested this example.

space. Then, we have $y_1 y_3 - y_2^2 = 0$. Thus, $L^{\circledS 2}$ has order 5 (instead of 6) and one can check that the Galois group has an invariant of degree 2. The corresponding polynomial first integral is $-2yy'' + (y')^2 - 4ry^2$ (for some values of $r$, this result appears in the literature as the "Brioschi identity", see [13]). This is a case where our techniques for degeneracies apply.

### 5.2 The Hurwitz system

Consider the Hurwitz system $Y' = AY$ with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{\frac{792}{343}x - \frac{40805}{24696}}{x^2(x-1)^2} & -\frac{72/7\,x^2 - \frac{2963}{252}x + 20/9}{x^2(x-1)^2} & -\frac{7x-4}{x(x-1)} \end{pmatrix}.$$

This system is irreducible. Singer and Ulmer have shown in [21] that it has Galois group $G_{168}$. From [19], we derive that it has an invariant of degree 4 and corrolary 17 proves that $(S^m(A^*))$ has a rational solution. We have used our implementation to treat this example. Using the cyclic vector $(0, \cdots, 0, 1)$, we produced an equation $\mathcal{L}_m$ which was found to admit the rational solution $f = x^8(x-1)^6$; from this, we obtained the Darboux polynomial $M$ equal to:

$$
\begin{aligned}
&{y_2}^4 + 2\,\frac{(7x-4)\,{y_2}^3\,y_1}{x(x-1)} + \frac{8(342x-49)\,{y_2}^3\,y_0}{441x^2(x-1)} + \frac{3\left(-2743x+784+2400x^2\right){y_2}^2\,{y_1}^2}{98x^2(x-1)^2} \\
&+\frac{\left(133920x^2-95735x+10976\right){y_2}^2\,y_1\,y_0}{2058x^3(x-1)^2} + \frac{\left(7464960x^3-9607401x^2+2292136x-153664\right){y_2}^2\,{y_0}^2}{518616(x-1)^3x^4} \\
&+\frac{\left(117504x^3-201457x^2+115166x-21952\right)y_2\,{y_1}^3}{686x^3(x-1)^3} \\
&+\frac{\left(3276288x^3-4215051x^2+1607956x-153664\right)y_2\,{y_1}^2\,y_0}{14406(x-1)^3x^4} \\
&+\frac{\left(-71659952x+365036544x^4-678527307x^3+380764774x^2+4302592\right)y_2\,y_1\,{y_0}^2}{3630312x^5(x-1)^4} \\
&+\frac{\left(-93927904x+1451188224x^4-2077051005x^3+713525139x^2+4302592\right)y_2\,{y_0}^3}{98018424x^6(x-1)^4} \\
&+\frac{\left(-4298112x+5750784x^4-13146820x^3+11273973x^2+614656\right){y_1}^4}{38416x^4(x-1)^4} \\
&+\frac{\left(-15004192x+45785088x^4-85081374x^3+56160371x^2+1229312\right){y_1}^3\,y_0}{172872x^5(x-1)^4} \\
&+(37585800183x^3-14153668240x^2+17844240384x^5-43373550816x^4+2215900736x \\
&-120472576){y_1}^2\,{y_0}^2/\left(101648736x^6(x-1)^5\right) \\
&+(1300959213471x^3-343699174616x^2+992612745216x^5-1988514598464x^4 \\
&+39750571840x-1686616064)y_1{y_0}^3/\left(19211611104x^7(x-1)^5\right) \\
&+(55168371523584x^6-142125993591552x^5+125442123785361x^4-45962050792788x^3 \\
&+8156158257856x^2-702269066240x+23612624896){y_0}^4/\left(9682651996416x^8(x-1)^6\right)
\end{aligned}
$$

and, of course, $fM$ is a polynomial first integral of the system. Now, we can easily obtain the other first integrals. It is classic (see e.g [25] pages 223-226) that the Hessian of an invariant is an invariant, the bordered Hessian of two invariants is an invariant, and the Jacobian of $n$ invariants is again an invariant (same for semi-invariants). With the isomorphism of theorem 12, one can check (see [27]) that this remains true for the Darboux polynomials. Thus, taking the Hessian of $M$, we obtain a Darboux polynomial $H(M)$ of degree 6 that yields a first integral; their bordered Hessian yields a first integral of degree 14 and the Jacobian of these three yields a first integral of degree 21. This way, we obtain all first integrals of the system because these four invariants generate (see[22]) the invariant ring of $G_{168}$. Note that, in this example, this strategy is absolutely necessary in practice because the size of the coefficients obtained from cyclic vectors of the symmetric power grows dramatically and the equations for degrees 14 and 21 are too big to be constructed (see e.g [27, 5]). This emphasizes another contribution of this paper: given a linear differential equation (or system), one can detect invariants by finding rational solutions of symmetric power but this does not give their expression; our construction gives us explicit expressions that allow symbolic manipulations on invariants or semi-invariants.

### 5.3 Some remarks and questions

• This example yields a first question. In [20], it is shown that, if the Galois group of a three-dimensional system is a proper irreducible subgroup of $SL_3(\mathcal{C})$, then it has a semi-invariant of degree at most 36. However, in a careful reading of their paper (and of the classical literature on invariants of finite groups), one finds that a semi-invariant of *minimal* degree has degree at most 6 (and an invariant has degree at most 12). It would be of interest to have such sharp bounds for higher values of $n$.

• Usually, when a differential equation is reducible, one factors the equation into equations of lower order. But, it is not clear then what the degrees of the semi-invariants are. For example, consider the equation

$$L(y) = (\partial^2 - x)(\partial^2 - x)(y) = y^{(4)} - 2xy'' - 2y' + x^2y = 0$$

Note that $(\partial^2 - x)(y)$ has Galois group $SL_2(\mathcal{C})$ and thus no first integral (see [6]). However, one can check (and theoretically prove, see [27]) that $L$ has a polynomial first integral of degree 2. So, an interesting question would be: how can one bound the degree of the semi-invariants in the case of a reducible system?

• In [17], Singer proves that a second order homogeneous linear differential equation has a liouvillian first integral if and only if it has a liouvillian solution. Our results show how this statement behaves for higher order equations:

**Proposition 18.** *Assume that a homogeneous linear differential equation $L$ has a reductive Galois group. If $L$ has a liouvillian solution, then it has a liouvillian first integral. The converse is false in general.*

*Proof.* If $L$ has a liouvillian solution, then it has a solution $z$ whose logarithmic derivative is algebraic (see [14]) and $G$ has a semi-invariant $z$; by theorem 12, we can deduce a Darboux polynomial $M_z$ from this semi-invariant, and then $zM_z$ is a liouvillian first integral of $L$. Conversely, it may occur that there is a semi-invariant but no liouvillian solution. For example, take the equation $y''-xy=0$. This equation has Galois group $SL(2,C)$. We form its symmetric square $L(y)=y'''-4xy'-2y=0$. This equation has Galois group $PSL(2,C)$ and no liouvillian solutions; however, we saw in section 5.1 that it has a polynomial first integral. □

• More developpements and applications of the results of this paper will be found in the author's PhD dissertation ([27]).

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