Self-equilibrated Functions in Dual Vector Spaces: a Boundedness Criterion

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reporting joint work with E. Ernst and M. Volle

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Credits

This talk is intended to focus on the question of giving criteria to ensure that an extended-real-valued function defined on a locally convex vector space is bounded below. It is based mainly on two papers:

- E. ERNST, M. THÉRA and M. VOLLE, *Work in progress*
- H. ATTOUCH and H. BRÉZIS, Duality for the sum of convex functions in general Banach spaces, *Aspects of mathematics and its applications*, Collect. Pap. Hon. L. Nachbin, 125-133, 1986.

Content of the presentation

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  - Self-equilibrated sets

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- Two questions

First criteria (☼)

Consider the relation $\inf_X \Phi = -\Phi^\star(0)$;

Φ is bounded below $\iff$
$0 \in \mathrm{Dom}\,\Phi^\star = \{x \in X^\star :\ \Phi^\star(x) < +\infty\}$ (the effective domain of $\Phi^\star$).

As $\mathrm{Dom}\,\Phi^\star$ is a convex set, relation $0 \in \mathrm{Dom}\,\Phi^\star$ occurs when $\mathbb{R}_+ \mathrm{Dom}\,\Phi^\star = \{\lambda x :\ \lambda \geq 0,\ x \in \mathrm{Dom}\,\Phi^\star\}$ (the cone spanned by the effective domain of $\Phi^\star$) is a **linear subspace** of $X^\star$.

(☼) an extended-real-valued function is bounded below on a locally convex space X when the cone spanned by the effective domain of its conjugate is a linear subspace of the topological dual $X^\star$ of X.

Another tool: the imfimal convolution

Another tool which can successfully be used to derive boundedness criteria is the concept of infimal convolution:

$$\Phi \Box \Psi(x) = \inf_{y \in X} \left(\Phi(x-y) + \Psi(y) \right) \ \forall x \in X$$

Given $\Phi, \Psi : X \to \mathbb{R} \cup \{+\infty\}$, $\Phi \Box \Psi$ is said to be exact at $x \in \operatorname{Dom} \Phi \Box \Psi$ provided the inf appearing in the definition is attained.

The infimal convolution of functions in the class $\Gamma_0(X)$ has been extensively studied due to its remarkable role in duality theory and in particular in the central problem of computing the conjugate of the sum of two functions. (Rockafellar, Moreau, ...)

Second criteria (▲)

Needs of a *qualification condition*, that is a condition stated in terms of $\Phi^\star$ and $\Psi^\star$ which ensures that relation

$$(\mathcal{QC}) \quad \Phi \Box \Psi = (\Phi^\star + \Psi^\star)^\star$$

holds and that the infimal convolution is exact.

Indeed, a $\Gamma_0(X)$-function is bounded below when some qualification condition is fulfilled by its conjugate and by the indicator function of the singleton $\{0\}$ (the conjugate of the constant function equal to zero).

(▲) a $\Gamma_0(X)$-function is bounded below (and achieves its infimum) over a locally convex space X when its conjugate is finite and continuous at 0.

A third criteria (◑)

This qualification condition has known numerous refinements. A significant step was accomplished when Attouch and Brézis proved for reflexive Banach spaces that a sufficient condition for relation $(\mathcal{QC})$ to hold and for the infimal convolution to be exact is that the convex cone $\mathbb{R}_+\left(\mathrm{Dom}\,\Phi^\star - \mathrm{Dom}\,\Psi^\star\right)$ spanned by the difference of the effective domains of $\Phi^\star$ and $\Psi^\star$ is a closed linear space. This result infers a new boundedness criterion:

(◑) a $\Gamma_0(X)$-function is bounded below (and reaches its infimum) over a reflexive Banach space X if the cone spanned by the effective domain of its conjugate is a closed linear space.

Continued

It can be observed that a $\Gamma_0(X)$- functional defined on a reflexive Banach space fulfills conditions of criterion (◑) $\iff$ if it is semi-coercive that is if there exists a closed subspace V of X such that

- $\Phi(x) = \Phi(x+v),\ \forall x \in X,\ \forall\, v \in V,$
- the quotient functional $\overline{\Phi} :\ X/V \to \mathbb{R} \cup \{+\infty\}$, defined by :

$$\overline{\Phi}(\overline{x}) = \Phi(x),\ \forall x \in \overline{x},\ \forall\, \overline{x} \in X/V,$$

is coercive, in the sense that $\left(\overline{\Phi}\right)^{-1}(] - \infty, r])$ is bounded for every $r \in \mathbb{R}$.

Continued

Every coercive functional is semicoercive. Let us also mention two of the most frequently encountered semicoercive functionals:

(i) The distance functional to a closed subspace F of an arbitrary Banach space X: $J_1(x) = \mathrm{dist}(x, F) = \inf_{y \in F} \|x - y\|$.

(ii) If $\Omega \subseteq^n$ is a bounded subset with a smooth boundary, and $H^1(\Omega)$ is the corresponding Sobolev space,

$$J_2 : H^1(\Omega) \to \mathbb{R},\ J_2(u) = \int_\Omega |\nabla u(x)|^2 \, dx.$$

Combine two remarks

Remark 1- A well-known result says that on every nonreflexive Banach space one may find a coercive $\Gamma_0(X)$-functional Φ which does not achieve its infimum.

Remark 1- Recall that the coerciveness of Φ implies that 0^* is an interior point of $\mathrm{Dom}\,\Phi^*$, which thus spans X^*.

Criterion (◑) does no longer work in nonreflexive Banach spaces

A fourth criteria : (◐)

Let us remark that, when seen as a boundedness criterion, (◑) and (▲) are mere corollaries of (☼). Indeed, criterion (◑) asks for the continuity of the conjugate at 0. This forces the cone spanned by its effective domain to coincide with $X^\star$, while criterion (▲) directly imposes to this cone to be a closed linear subspace of $X^\star$.

Developing ideas going back to Joly, a new qualification condition was introduced by Moussaoui and Volle. This qualification condition leads to

(◐) a $\Gamma_0(X)$-function is bounded below (and achieves its infimum) over a locally convex space X provided the closure of the cone spanned by the effective domain of its conjugate is a linear space and all its sublevel sets are locally weakly compact.

When X is reflexive, take a **closed, convex and unbounded** set C with half-lines (linearly bounded =$C_\infty = \{0\}$) and consider the gauge function given by

$$f_C(x) = \frac{1}{\sup\{\lambda \geq 0 \,:\, \lambda x \in C\}}, \text{ where } \frac{1}{0} := +\infty.$$

$f_C \in \Gamma_0(X)$ **and is not concerned by criteria (◑) and (◐).**

- Criterion (◑) does not concern f_C:

  Since $\mathbb{R}_+\mathrm{Dom}\, f_C^* = \mathcal{B}(C)$, then $\mathbb{R}_+\mathrm{Dom}\, f_C^*$ is a dense (since $C^\infty = \{0\}$) and a proper (as C is unbounded) subset of X^*.

- Criterion (◐) does not concern f_C:

  C is one of the level sets of f_C, and cannot be locally weakly compact.

A fifth criteria : (▼)

Our aim in this presentation is to adopt a totally different standpoint in order to derive new criteria for extended-real-valued functions to be bounded below.

As an example, let a us start with a well known argument. When X is locally convex, a $\Gamma_0(X)$-function achieves its infimum on each weakly compact convex sets of X; therefore it is bounded below on such a set. **Accordingly, a $\Gamma_0(X)$- function is bounded below if one of its sublevel sets is weakly compact**. When the underlying space X is a reflexive Banach space, this reads as follows:

(▼) a $\Gamma_0(X)$-function is bounded below on a reflexive Banach space X if at least one of its sublevel sets is bounded.

A fifth criteria : (▼)

This is not a very interesting result in itself (it is an easy application of either of criteria (▲) and (◑)).

However, the method used in deriving (▼) may lead to more comprehensive results.

Indeed, the lack of generality of criterion (▼) is obviously a consequence of the fact that the set of bounded closed and convex sets does not exhaust, and by far, the class of sets which cannot be sublevel sets of $\Gamma_0(X)$ functions that are unbounded below.

Clearly, in oder to state a nontrivial boundedness criterion of type (▼), one must first completely characterize the class of those sets which cannot be sublevel sets of functions unbounded below.

Needs of a short review on basic notations

The recession cone to the closed convex set S is the closed convex cone S^∞ defined as

$$S^\infty = \{v \in X : \;\; \forall\, \lambda > 0,\, \forall\, x_0 \in S,\; x_0 + \lambda v \in S\}\,.$$

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S is called linearly bounded whenever $S^\infty = \{0\}$.
Given a closed convex subset S of X, the domain of the support function given by

$$\sigma_S(f) := \sup_{x \in S} \langle f, x \rangle$$

is the barrier cone of S:

$$\mathcal{B}(S) = \{f \in X^* : \sigma_S(f) < +\infty\} = \mathrm{Dom}\, \sigma_S.$$

Needs of a short review on basic notations

The recession function Φ_∞ of Φ is given by

$$\Phi_\infty(x) = \sup_{t>0} \frac{\Phi(x_0 + tx) - \Phi(x_0)}{t}.$$

x_0 arbitrary in X.

$\Gamma_0(X)$ is the set of lower semicontuniuous convex proper (not identically equal to $+\infty$) functions $\Phi : X \to \mathbb{R} \cup \{+\infty\}$.

Setting

The characterization of the class of those sets which cannot be sublevel sets of functions unbounded below will be achieved in the general setting of dual vector spaces, say X and Y, where the duality bracket $\langle \cdot, \cdot \rangle : X \times Y \to \mathbb{R}$ satisfies:

$$\forall x \neq 0,\ x \in X,\ \exists y \in Y \text{ s.t. } \langle x, y \rangle \neq 0,$$

$$\forall y \neq 0,\ y \in Y,\ \exists x \in X \text{ s.t. } \langle x, y \rangle \neq 0.$$

A locally convex topology on X (respectively on Y) is called compatible with the duality if for that topology the maps of type $\langle \cdot, y \rangle : X \to \mathbb{R}$, $y \in Y$ (respectively $\langle x, \cdot \rangle : Y \to \mathbb{R}$, $x \in X$) are the only linear continuous functions on X (respectively on Y). Y may be identified with the topological dual $X^\star$ of X (and X with $Y^\star$).

Self-equilibrated sets

The main tool used is the notion of self-equilibrated set

We say that a subset S of the real Hausdorff topological vector space X is *self-equilibrated* if every linear and continuous function which is bounded from above on S is also bounded below on the same set.

Equivalently, S is self-equilibrated if $\mathcal{B}(S) = -\mathcal{B}(S)$, that is if the barrier cone $\mathcal{B}(S)$ is a linear subspace of X^*.

A subset S of X is self-equilibrated if and only if its closed convex hull $\overline{\mathrm{co}}(S)$ is self-equilibrated. As a result we will focus our attention on the study of closed convex self-equilibrated sets.

A first boundedness criterion

Follows from Proposition ☎

Proposition ☎ a set S may play the role of a sublevel set of an unbounded below extended-real-valued function minorized on S by some affine mapping, if and only if S it is not self-equilibrated.

and reads as follows

Theorem ☺ Every extended-real-valued function minorized on a self-equilibrated sublevel set by an affine mapping is bounded below.

Two applications of Theorem ☻

- to a subclass of quasi-convex functions of the type

$$\Psi(x) = \sup_{i\in I} \left(\min(\langle\cdot, y_i\rangle + a_i, r_i)\right), \qquad (1)$$

where $y_i \in Y$, $a_i, r_i \in \mathbb{R}$, while I is an arbitrary index set.

is bounded below provided one of its sublevel sets is a self-equilibrated proper subset of X.

- the second one is a specification of Theorem ☻ to the particular case of functions minorized by an affine mapping on the whole underlying space X:

Theorem ✠ An extended-real-valued function minorized by an affine mapping is bounded below provided at least one of its sublevel sets is self-equilibrated.

A second boundedness criterion

We propose a new boundedness criterion, valid as Theorem ✠, only for functions which are minorized on the whole space X by some affine map. It relies on the notion of self-equilibrated function. Φ is self-equilibrated provided that if it grows faster than a linear map $\langle\cdot, y\rangle$, $y \in Y$,

$$\left[\sup_{x \in S} \Phi(x) < +\infty\right] \Longrightarrow \left[\sup_{x \in S} \langle x, y\rangle < +\infty\right]$$

(such y is called a growth direction for Φ) then the same holds for the opposite linear map, $\langle\cdot, -y\rangle$.

Theorem 🔔 An extended-real-valued function is bounded below provided it is self-equilibrated and minorized on X by an affine mapping.

Properties of the cone G_Φ of the growth directions of Φ

Lemma The element $y \in Y$ is a growth direction for the function Φ if and only if the mapping $\langle \cdot, y \rangle$ is bounded above on every sublevel set of Φ.

G_Φ can be expressed in terms of the barrier cones of the sublevel sets of f.

$$G_\Phi = \bigcap_{t \in \mathbb{R}} \mathcal{B}([\Phi \leq t]).$$

$$\boxed{\mathcal{B}(\mathrm{Dom})\Phi \subset \mathcal{B}([\Phi \leq t]) \implies \mathcal{B}(\mathrm{Dom}\,\Phi) \subset G_\Phi.}$$

and also

$$\boxed{\mathbb{R}_+ \,\mathrm{Dom}\,\Phi^* \subset G_\Phi}$$

no equality in general

In general, the set of growth direction is larger than the cone spanned by the effective domain of the conjugate. Indeed, consider the real function

$$\Phi : \mathbb{R} \longrightarrow \mathbb{R},\ \Phi(x) = \frac{x - |x|}{2};$$

the effective domain of its conjugate is void, while the set of growth direction is the half-line $G_\Phi = \{x \in \mathbb{R} :\ x \geq 0\}$.

@@@

Proposition Let Φ be an extended-real-valued function $\mathrm{Dom}\,\Phi^\star \neq \emptyset$. Then, for any vector space topology on Y, compatible or not with the duality, then

$$\mathcal{B}(\mathrm{Dom}\,\Phi) \subset \overline{\mathbb{R}_+ \mathrm{Dom}\,\Phi^\star}.$$

Proposition Let Φ be a convex extended-real-valued function with $\mathrm{Dom}\,\Phi^\star \neq \emptyset$. Then, for any vector space topology on Y, compatible or not with the duality, then

$$\overline{G_\Phi} = \overline{\mathbb{R}_+ \mathrm{Dom}\,\Phi^\star}.$$

When Φ is convex

Lemma The set of the growth directions for any convex function $\Phi \in \Gamma_0(X)$ is the union between the cone spanned by the effective domain of Φ^* and the barrier cone of the effective domain of Φ:

$$G_\Phi = \mathbb{R}_+ \mathrm{Dom}\Phi^* \cup \mathcal{B}(\mathrm{Dom}\Phi).$$

A consequence

Let us describe two remarkable cases where the set of growth directions and the cone spanned by the effective domain of the conjugate coincide.

Let Φ be a real-valued convex function. Then either

$$G_\Phi = \mathbb{R}_+(\mathrm{Dom}\,\Phi^\star) \;\text{ or }\; G_\Phi = \{0\}.$$

The second one concerns convex functions which are bounded below:

For any convex extended-real-valued function Φ bounded below one has

$$G_\Phi = \mathbb{R}_+\left(\mathrm{Dom}\,\Phi^\star\right). \tag{2}$$

Remark

The following result makes the connection between our second boundedness result, Theorem 🔔 and the criteria (☼) and (◑).

Proposition- Let Φ be a convex extended-real-valued function minorized by an affine mapping on the whole underlying space X. The following are equivalent:

1. Φ is self-equilibrated;
2. $\mathbb{R}_+ \left(\operatorname{Dom} \Phi^\star\right)$ is a linear subspace of Y.

Remark

When applied to the duality between X (supposed to be locally convex) and $X^\star$(topological dual), and using the fact that in this case, the effective domain of the conjugate of every proper lower semi-continuous function is nonempty, Theorem ♣ infers:

Theorem ♣♣ Every proper lsc function Φ defined on a locally convex vector space X is bounded below if the set of all linear and continuous functions $f : X \longrightarrow \mathbb{R}$ such that

$$[\sup_A \Phi < +\infty] \Rightarrow [\sup_A f < +\infty]$$

is a linear space.

Two final remarks

Our two criteria (Theorem ☻and Theorem ♠) **assign to the infimum of the function to be finite, but do not garantee to the function to achieve its infimum**

Does a self-equilibrated lower semi-continuous convex function reaches its infimum ?

Answer : **YES**

Suppose that X is the algebraical dual of Y (that is all the linear maps on Y are of form $\langle x, \cdot \rangle$, $x \in X$). Then every self-equilibrated $\Gamma_0(X)$-function achieves its infimum on X.

Answer **NO**

On every Banach vector space X of infinite dimension there is a self-equilibrated $\Gamma_0(X)$-function which does not achieve its infimum

Thank you !