

Sur le problème d'évolution associé à l'équation de Monge-Kantorovich

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Chambéry, 25 Avril 2008

Monge-Kantorovich evolution equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain, $N \geq 1$, $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$

$$(EMK) \quad \left\{ \begin{array}{ll} \frac{\partial h}{\partial t}(x, t) - \nabla \cdot m(x, t) \nabla h(x, t) = \mu & \text{in } Q \\ m(x, t) \geq 0, |\nabla h(x, t)| \leq 1 & \text{in } Q \\ m(1 - |\nabla h(x, t)|) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega \end{array} \right.$$

Main interest :

- 1 Existence and uniqueness of a solution : μ a Radon measure
- 2 Numerical analysis : representation of $t \in [0, T) \rightarrow (u(t, x), m(t, x) \nabla u(t, x))$.
- 3 Large time behavior, as $t \rightarrow \infty$.

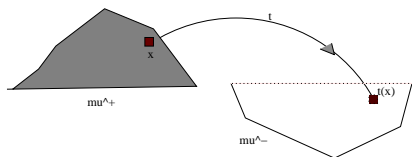
Motivation I : optimal mass transportation

Monge optimal mass transfer [Monge-1780] : Given two measures μ^+ and μ^- in \mathbb{R}^N , such that $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$ and $\text{spt}(\mu^+) \neq \text{spt}(\mu^-)$

$$\text{Min}_{t \in \mathcal{A}} \int_{\mathbb{R}^N} |x - t(x)| d\mu^+(x)$$

$$\mathcal{A} = \left\{ t : \text{spt}(\mu^+) \rightarrow \text{spt}(\mu^-); \mu^+_{\#t} = \mu^- \right\}$$

i.e. $\mu^-(B) = \mu^+(t^{-1}(B))$



Monge-Kantorovich problem [Kantorovich-1940] :

- **Relaxed variant** : optimal plan transport

$$\bar{\mu} \in \Pi(\mu^+, \mu^-) := \left\{ \gamma \in \mathcal{M}_b(\mathbb{R}^N \times \mathbb{R}^N), \text{proj}_x(\gamma) = \mu^+, \text{proj}_y(\gamma) = \mu^- \right\}$$

$$\mathcal{J}(\bar{\mu}) = \min_{\gamma \in \Pi(\mu^+, \mu^-)} \mathcal{J}(\gamma) \quad \mathcal{J}(\gamma) = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x - y| d\gamma(x, y)$$

Is $\bar{\mu}$ supported in a graph? i.e. $\exists ? t \in \mathcal{A}, \bar{\mu}(E) = \mu^+(\{x \in \mathbb{R}^N; (x, t(x)) \in E\})$

- **Dual variational principle** : Shipper (principe du convoyeur) $\bar{\mu}$ is optimal if and only if

$$\mathcal{J}(\bar{\mu}) = \max_{\xi \in \text{Lip}_1(\mathbb{R}^N)} \left\{ \int_{\mathbb{R}^N} \xi d\mu^+ + \int_{\mathbb{R}^N} \xi d\mu^- \right\} = \int_{\mathbb{R}^N} \bar{u} d\mu^+ + \int_{\mathbb{R}^N} \bar{u} d\mu^- .$$

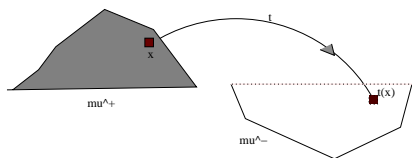
- *Memoire sur la théorie de Déblais et des remblais*, Gaspard Monge, Histoire de l'Académie des Sciences de Paris, 1781.
- *On the transfer of masses*, L.V. Kantorovich, Dokl. Acad. Nauk. SSSR 37, 227-229 (1942).

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$$\text{Min}_{t \in \mathcal{A}} \int_{\mathbb{R}^N} |x - t(x)| d\mu^+(x)$$

$$\mathcal{A} = \left\{ \begin{array}{l} t : \text{spt}(\mu^+) \rightarrow \text{spt}(\mu^-) ; \mu^+_{\#t} = \mu^- \\ \text{i.e. } \mu^-(B) = \mu^+(t^{-1}(B)) \end{array} \right\}$$



Evans and Gangbo [1999] : **How to fashion an optimal transport t ?**

Monge-Kantorovich equation (Stationary equation of (EMK))

$$(MK) \quad \left\{ \begin{array}{l} -\nabla \cdot (m \nabla u) = \mu \quad (:= \mu^+ - \mu^-) \\ m \geq 0, |\nabla u| \leq 1, m(|\nabla u| - 1) = 0 \end{array} \right\} \implies m \nabla u : \text{the flux transportation}$$

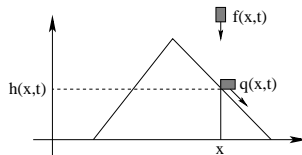
- m gives the transport density
- $-\nabla u$ gives the direction of the optimal transportation

Remark : If (u, m) solves (MK) then u is the Kantorovich potential ; i.e.

$$\max_{\xi \in \text{Lip}_1(\mathbb{R}^N)} \int_{\mathbb{R}^N} \xi d\mu = \int_{\mathbb{R}^N} u d\mu$$

Motivation II : Prigozhin model for growing sandpile

Motivation II : Prigoghin model for growing sandpile



- Conservation of mass :

- The flow of the granular material is confined in a thin boundary layer moving down the slopes of a growing pile
- The density of the material is constant

$$\frac{\partial h}{\partial t}(x, t) = -\nabla \cdot q(x, t) + \mu(x, t)$$

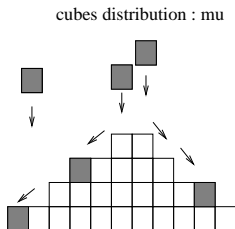
- Surface flow directed by the steepest descent :

$$\exists m = m(x, t) \geq 0 \text{ such that } q(x, t) = -m(x, t) \nabla h(x, t)$$

- No pouring over the parts of the pile surface inclined less than α :

$$|\nabla h(x, t)| < \gamma \implies m(x, t) = 0$$

Motivation III : discrete model (stochastic)



The associated continuous model :

N cubes of sides $1/N$, $N \rightarrow \infty \Rightarrow$ Evolution Monge-Kantorovich equation

Works in progress :

- Some continuous model associated with the Stochastic approach [Poaty-Ig].
- Stability of protective sea dyke (Projet FLUPARTI soutenue et financé par le conseil régional de Picardie)
- *A stochastic model for growing sandpiles and its continuum limit*, L. C. Evans, F. Rezakhanlou, Comm. Math. Phys., 197 (1998), no2, 325-345.

Boundary condition : Dirichlet

- If $\mu^+(\mathbf{R}^N) = \mu^-(\mathbf{R}^N)$ and μ^+, μ^- are supported in bounded domain : by taking Ω large enough we can assume that $u = 0$ on $\partial\Omega$;
- If $\mu^+(\mathbf{R}^N) \neq \mu^-(\mathbf{R}^N)$ and μ^+, μ^- are supported in bounded domain, then one needs :
 - There exists Γ such that $\mathcal{H}^{N-1}(\Gamma) = 0$ and replace the euclidean $c(x, y) = |x - y|$ by the semi-geodesic distance taking into account the boundary; i.e.

$$c_{\Gamma}(x, y) = \min \left(|x - y|, \text{dist}(x, \Gamma) + \text{dist}(y, \Gamma) \right).$$

- **Dirichlet Monge-Kantorovich problem** (Bouchité, Buttazzo, De Pascale ...)

$$\inf \left\{ \iint c_{\Gamma}(x, y) d\nu(x, y) ; \nu \in \mathcal{M}(\Omega \times \Omega) \text{ s.t. } \pi_{\#}^1 \nu - \pi_{\#}^2 \nu = \mu_1 - \mu_2 \right\}$$

Monge-Kantorovich evolution equation with Dirichlet boundary condition

$$(EMK) \quad \left\{ \begin{array}{ll} \frac{\partial h}{\partial t} - \nabla \cdot (m \nabla h) = \mu & \text{in } Q \\ m \geq 0, |\nabla h| \leq 1 & \text{in } Q \\ m(1 - |\nabla h|) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega \end{array} \right.$$

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- 3 Large time behavior, as $t \rightarrow \infty$.

Références de l'exposé

<http://www.mathinfo.u-picardie.fr/igbida/>



S. Dumont and N. Igbida,

Back on a Dual Formulation for the Growing Sandpile Problem, *submitted*.



N. Igbida,

Evolution Monge-Kantorovich Equation, *submitted*.



N. Igbida,

On Monge-Kantorovich Equation, *Preprint*.

Main Difficulties

$$\left\{ \begin{array}{l} \partial_t u - \nabla \cdot (m \nabla u) = \mu \text{ in } \mathcal{D}'(\Omega) \\ |\nabla u| \leq 1, m \geq 0, m(|\nabla u| - 1) = 0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} u_t + \partial \mathbf{I}_K(u) \ni \mu, \\ K = \{z \in H_0^1(\Omega) ; |z| \leq 1 \text{ in } \Omega\} \end{array} \right.$$

i.e. $\forall t \in [0, T), u(t) \in K$ and $\int_{\Omega} u(t) (\mu(t) - \partial_t u(t)) = \max_{\xi \in K} \int (\mu(t) - \partial_t u(t)) \xi$.

Nonlinear semi-group theory \Rightarrow existence and uniqueness of "variational" solution u

Main Difficulties

- The main information are in the flux $\Phi = m \nabla u$:

Monge-Kantorovich	\longrightarrow	flux transport
Sandpile	\longrightarrow	numerical analysis

- In general, m is not regular : m is a measure.
- How to define the solution for (MK) and (EMK) : m is a measure and $\nabla u \in L^\infty(\Omega)$.
- How to do numerical analysis for (MK) and (EMK) : regularisation technics are not stable.

Notion of solution : how to define $m \nabla u$ (m a Radon measure)

Stationary problem : Theory of tangential gradient of u with respect to Radon measure (cf. Bouchitté, Buttazzo, De Pascale, Seppecher ...) :

$$(MK) \quad \begin{cases} -\nabla \cdot (m \nabla_m u) = \mu & \text{in } \mathcal{D}'(\Omega) \\ |\nabla_m u| = 1 & m - \text{ a.e. in } \Omega. \end{cases} \quad (\text{Monge-Kantorovich equation})$$

In particular :

- Let $\nu \in \mathcal{M}_b(\Omega)$ and $1 < p < \infty$

$$w = \nabla_\nu u \in L^p_\nu(\Omega) \Leftrightarrow \exists u_h \in C^1(\Omega), (u_h, \nabla u_h) \rightarrow (u, w) \quad \text{in } L^p_\nu(\Omega)^{N+1}.$$

- For any $F \in L^{p'}_\nu(\Omega)^N$, such that $-\nabla \cdot (F) = \mu \in L^p(\Omega)$, we have

$$\int_\Omega \nabla_\nu u \cdot F \, d\nu = - \int_\Omega u \mu$$

for any $u \in \mathcal{C}_0(\Omega)$ such that u is Lipchitz.

Existence, uniqueness of $m \nabla_m u$, regularity, related transport for Monge problem... : Ambrosio, Bouchitté, Buttazzo, De Pascale, Evans, Seppecher

Notion of solution : how to define $m \nabla u$ (m a Radon measure)

Evolution problem :

- Theory ? tangential derivative for time dependent functions : $t \in (0, T) \rightarrow \nabla_\nu u(t)$.
- For $\mu \in L^2(Q)$: existence, uniqueness and numerical analysis by dual formulation
 - Prigozhin - 1996 : $m \in (L^\infty(Q))^*$?

$$\begin{cases} \iint_Q (u_t - \mu) \xi + \langle m, \nabla u \cdot \nabla \xi \rangle = 0 \\ \langle m, \xi \rangle \geq 0, \quad \langle m, |\nabla u|^2 - 1 \rangle = 0 \end{cases}$$

- Prigozhin and Barrett - 2006 : $m \nabla u =: \Phi \in \mathcal{M}_b(Q)^N, \nabla \cdot \Phi \in L^2(Q)$.

- Minimizing problem related to Φ and $\int_0^t \Phi$ in $\mathcal{M}_b(Q)^N$ gives Φ and $u(t) = u_0 + \int_0^t f(s) ds + \nabla \cdot \int_0^t \Phi$.
- Equivalent formulation

$$\begin{cases} u_t - \nabla \cdot \Phi = \mu & \text{in } \mathcal{D}'(Q) \\ |\Phi|(Q) = \iint_Q u \nabla \cdot \Phi \end{cases}$$

- **Our aim** : a general theory of existence and uniqueness, large time behavior and numerical analysis.

Notion of solution : a simple characterization of the solution

Remark

Assume that $\Phi : \Omega \rightarrow \mathbb{R}^N$ and $u : \Omega \rightarrow \mathbb{R}$ are regular and $|\nabla u| \leq 1$ in Ω . Then

$$\begin{array}{l} \updownarrow \\ \bullet \quad m(|\nabla u| - 1) = 0 \quad \text{and} \quad \Phi = m \nabla u \quad \text{in} \quad \Omega \\ \bullet \quad m = |\Phi| = \Phi \cdot \nabla u \quad \text{in} \quad \Omega \\ \bullet \quad m = |\Phi| \quad \text{and} \quad \int_{\Omega} |\Phi| \leq \int_{\Omega} \Phi \cdot \nabla u \end{array}$$

Theorem (Ig,2007)

Let μ be a Radon measure and $u \in K$. Then,

$$\int_{\Omega} u \, d\mu = \max_{\xi \in K} \int_{\Omega} \xi \, d\mu \Leftrightarrow \exists \Phi \in \mathcal{M}_b(\Omega)^N \quad \begin{cases} -\nabla \cdot (\Phi) = \mu & \text{in } \mathcal{D}'(\Omega) \\ |\Phi|(\Omega) \leq \int_{\Omega} u \, d\mu \end{cases} \quad \text{(weak solution)}$$

$$\begin{array}{l} \Rightarrow \\ \Leftarrow \end{array} \quad \begin{cases} -\nabla \cdot (\Phi) = \mu & \text{in } \mathcal{D}'(\Omega) \\ \Phi = |\Phi| \nabla_{|\Phi|} u \end{cases} \quad \begin{array}{l} \Rightarrow \\ \Leftarrow \end{array} \quad \int_{\Omega} u \, d\mu = |\Phi|(\Omega) = \min\{|\nu|(\Omega) ; -\nabla \cdot \nu = \mu\}.$$

Remark

- If (u, Φ) is a weak solution and $\Phi \in L^1(\Omega)^N$, then $\Phi = |\Phi| \nabla u$ a.e. in Ω .
- The expression $|\Phi|(\Omega) = \int_{\Omega} u \, d\mu = \langle \nabla \cdot \Phi, u \rangle$ is a weak formulation of $|\Phi| = \Phi \cdot \nabla u$.

Equivalence for (MK) : main ideas of the proof

Lemma

Let $u \in W_0^{1,\infty}(\Omega)$ be s.t. $|\nabla u| \leq 1$ a.e. in Ω . Then,

- for any $\nu \in \mathcal{M}_b(\Omega)^+$,

$$|\nabla_\nu u| \leq 1 \quad \mu - \text{a.e. in } \Omega.$$

- there exists $z_\varepsilon \in K \cap \mathcal{D}(\Omega)$ such that $z_\varepsilon \rightarrow z$ in $C_0(\Omega)$.

- (u, Φ) is a weak solution $\implies |\Phi|(\Omega) = \int_\Omega u \, d\mu$ and u is a variational solution :

- $\int_\Omega u \, d\mu = \lim_{\varepsilon \rightarrow 0} \int_\Omega \nabla u_\varepsilon \cdot \frac{\Phi}{|\Phi|} \, d|\Phi| \leq |\Phi|(\Omega)$, where $\frac{\Phi}{|\Phi|} \, d|\Phi|$ is the polar representation of Φ .

- $\int_\Omega \xi \, d\mu = \int_\Omega \frac{\Phi}{|\Phi|} \cdot \nabla \xi \, d|\Phi| \leq |\Phi|(\Omega) = \int_\Omega u \, d\mu$.

- (u, Φ) is a weak solution $\implies (u, \Phi)$ is a solution of (MK) :

$$\begin{cases} |\Phi|(\Omega) = \int_\Omega u \, d\mu = \int_\Omega \nabla_{|\Phi|} u \cdot \frac{\Phi}{|\Phi|} \, d|\Phi| \\ |\nabla_{|\Phi|} u| \leq 1 \quad |\Phi| - \text{a.e. in } \Omega \end{cases} \implies \nabla_{|\Phi|} u \cdot \frac{\Phi}{|\Phi|} = 1 \quad |\Phi| - \text{a.e. in } \Omega.$$

- (u, Φ) satisfies (MK) $\implies (u, \Phi)$ is a weak solution

$$|\Phi|(\Omega) = \int_\Omega |\nabla_{|\Phi|} u|^2 \, d|\Phi| = \int_\Omega \frac{\Phi}{|\Phi|} \cdot \nabla_{|\Phi|} u \, d|\Phi| = \int_\Omega u \, d\mu$$

- u is a variational solution $\implies \exists \Phi$ such that (u, Φ) is a weak solution : $p \rightarrow \infty$ in $\Delta_p u_p = \mu$.

Notion of solution : Evolution Monge-Kantorovich equation

- $L^q(0, T; w^*-\mathcal{M}_b(\Omega)) = \left\{ \psi : (0, T) \rightarrow \mathcal{M}_b(\Omega), \text{ weak}^* \text{ measurable ; } \int_0^T |\psi(t)|^q(\Omega) < \infty \right\}$.
- $BV(0, T; w^*-\mathcal{M}_b(\Omega)) = \left\{ \mu \in L^1(0, T; w^*-\mathcal{M}_b(\Omega)) ; \limsup_{h \rightarrow 0} \frac{1}{h} \int_0^{T-h} |\mu(\tau+h) - \mu(\tau)|(\Omega) d\tau < \infty \right\}$.
- In Q the condition $|\Phi| = \Phi \cdot \nabla u$ may be represented by one of the following formulation :
 - $|\Phi(t)|(\Omega) \leq \left\langle \nabla \cdot \Phi(t), u(t) \right\rangle = \left\langle \mu(t) - \partial_t u(t), u(t) \right\rangle$ for $t \in (0, T)$
 - $|\Phi|(Q) \leq \int_0^T \left\langle \nabla \cdot \Phi, u \right\rangle = \int_0^T \left\langle \mu(t) - \partial_t u(t), u(t) \right\rangle$

Proposition (Ig,2007)

If $\Phi \in L^1(0, T; w^*-\mathcal{M}_b(\Omega))^N$ and $\nabla \cdot \Phi \in L^1(0, T; w^*-\mathcal{M}_b(\Omega))$, $u \in \mathcal{C}([0, T]; L^2(\Omega)) \cap K_T$, $u(0) = u_0$, $\partial_t u - \nabla \cdot \Phi = \mu$ in $\mathcal{D}'(Q)$, then

$$\begin{array}{l} \updownarrow \\ \bullet \quad |\Phi(\cdot)|(\Omega) \leq -\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2(\cdot) + \int_{\Omega} u(\cdot) d\mu(\cdot) \quad \text{in } \mathcal{D}'(0, T) \\ \bullet \quad |\Phi|(Q) \leq \iint_Q u d\mu - \frac{1}{2} \int_{\Omega} u(T)^2 + \frac{1}{2} \int_{\Omega} u_0^2. \end{array}$$

Notion of solution : Evolution Monge-Kantorovich equation

Theorem (Ig,2007)

For any $u_0 \in K$ and $\mu \in BV(0, T, w^*-\mathcal{M}_b(\Omega))$, the problem (EMK) has a weak solution $(u, \Phi) : u \in C([0, T]; L^2(\Omega)) \cap K_T$, $\partial_t u \in L^\infty(0, T; w^*-\mathcal{M}_b(\Omega))$, $\Phi \in L^\infty(0, T; w^*-\mathcal{M}_b(\Omega)^n)$, $u(0) = u_0$, $\partial_t u - \nabla \cdot \Phi = \mu$ in $\mathcal{D}'(Q)$, and (u, Φ) satisfies

$$\begin{aligned} & \bullet \quad |\Phi(\cdot)|(\Omega) \leq -\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2(\cdot) + \int_{\Omega} u(\cdot) d\mu(\cdot) \quad \text{in } \mathcal{D}'(0, T) \\ & \bullet \quad |\Phi|(Q) \leq \iint_Q u \sigma d\mu - \frac{1}{2} \int_{\Omega} u(T)^2 + \frac{1}{2} \int_{\Omega} u_0^2. \end{aligned}$$

Moreover, u is the unique variational solution.

Theorem (Ig,2007)

For any $\mu \in L^1(0, T; w^*-\mathcal{M}_b(\Omega))$ and $u_0 \in K$, there exists (u, Φ) a weak solution of (EMK) i.e. $u \in C([0, T]; L^2(\Omega)) \cap K_T$, $u(0) = u_0$, $\Phi \in \mathcal{M}_b(Q)^n$, $\partial_t u - \nabla \cdot \Phi = \mu$ in $\mathcal{D}'(Q)$ and

$$\sup_{\eta \in C_0(\Omega)^N, \|\eta\|_\infty \leq 1} \iint_Q \sigma \frac{\Phi}{|\Phi|} \cdot \eta d|\Phi| \leq \frac{1}{2} \int_0^T \int_{\Omega} u^2 \sigma_t + \int_0^T \int_{\Omega} u \sigma d\mu \quad (1)$$

for any $\sigma \in \mathcal{D}(0, T)$. Moreover, u is the unique variational solution.

Remark

Numerical approximation

Assume that $\mu = f \in L^2(0, T; L^2(\Omega))$.

Time discretisation : Euler implicit schema.

- $0 < t_0 < t_1 < \dots < t_n = T$ with $t_i - t_{i-1} = \varepsilon \quad \forall i = 1, \dots, n$
- $f_1, \dots, f_n \in L^2(\Omega)$ such that

$$\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_{L^2(\Omega)} \leq \varepsilon.$$

- consider the approximation of $u(t)$ by :

$$u_\varepsilon(t) = \begin{cases} u_0 & \text{if } t \in]0, t_1] \\ u_i & \text{if } t \in]t_{i-1}, t_i] \end{cases} \quad i = 1, \dots, n$$

where u_i is given by :

$$u_i + \partial \mathbf{I}_K(u_i) = \varepsilon f_i + u_{i-1}$$

for $i \geq 1$ with $u_{i=0} = u_0$.

- u_ε is the ε -approximate solution of (P_μ) .

Numerical approximation : convergence of the ε -approximate solution

Nonlinear semigroup theory (in $L^2(\Omega)$)

↓ ↓ ↓ ↓ ↓

Theorem

Let $u_0 \in L^2(\Omega)$ be such that $|\nabla u_0| \leq 1$ and $f \in BV(0, T; L^2(\Omega))$. Then,

- 1 For any $\varepsilon > 0$ and any ε -discretisation, there exists a unique ε -approximation u_ε of (P_f) .
- 2 There exists a unique $u \in C([0, T]; L^2(\Omega))$ such that $u(0) = u_0$, and

$$\|u - u_\varepsilon\|_{C([0, T]; L^2(\Omega))} \leq C(\varepsilon) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

- 3 The function u given by 2. is the unique variational solution of (P_μ) .

Numerical approximation : a generic problem

The generic problem is

$$v + \partial I_K(v) = g$$

$$\Downarrow \Uparrow \quad \Downarrow \Uparrow \quad \Downarrow \Uparrow \quad \Downarrow \Uparrow$$

$$v = P_K g$$

where

- $K = \{z \in W^{1,\infty}(\Omega) \cap W_0^{1,2}(\Omega) ; |\nabla u| \leq 1\}$.
- P_K the projection onto the convex K , with respect to the $L^2(\Omega)$ norm :

$$v = P_K(u) \Leftrightarrow v \in K, \int_{\Omega} (v - u)(v - z) \geq 0 \quad \text{for any } z \in K.$$

Question :

For a given $g \in L^2$, how to compute $v = P_K g$?

Numerical approximation : Dual and Primal formulation

- $v = P_K g \iff J(v) = \min_{z \in K} J(z) ; J(z) = \frac{1}{2} \int_{\Omega} |z - g|^2.$

- We denote by

$$H_{div}(\Omega) = \{\sigma \in (L^2(\Omega))^N ; \operatorname{div}(\sigma) \in L^2(\Omega)\}$$

and

$$G(\sigma) = \frac{1}{2} \int_{\Omega} (\operatorname{div}(\sigma))^2 + \int_{\Omega} g \operatorname{div}(\sigma) + d \int_{\Omega} |\sigma|.$$

Theorem (Dumont-Ig,2007)

Let $g \in L^2(\Omega)$ and $v = P_K(g)$. Then, there exists a sequence $(w_{\epsilon})_{\epsilon > 0}$ in $H_{div}(\Omega)$, such that, as $\epsilon \rightarrow 0$,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \left(-G(w_{\epsilon}) \right) &= \sup_{w \in H_{div}(\Omega)} \left(-G(w) \right) \\ &= \min_{z \in L^2(\Omega)} J(z) \end{aligned}$$

and

$$\operatorname{div}(w_{\epsilon}) \rightarrow v - g \quad \text{in } L^2(\Omega).$$

Numerical approximation : main idea of the proof

- $\sup_{\sigma \in H_{\text{div}}(\Omega)} (-G(\sigma)) \leq \min_{z \in K} J(z)$.
- We consider the following elliptic equation

$$(S_\varepsilon) \quad \begin{cases} v_\varepsilon - \nabla \cdot w_\varepsilon = g & \text{in } \Omega \\ w_\varepsilon = \phi_\varepsilon(\nabla v_\varepsilon) & \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

where, for any $\varepsilon > 0$, $\phi_\varepsilon : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is given by

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon} (|r| - 1)^+ \frac{r}{|r|}, \quad \text{for any } r \in \mathbb{R}^N.$$

$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$

For any $g \in L^2(\Omega)$, (S_ε) has a unique solution v_ε , in the sense that $v_\varepsilon \in H_0^1(\Omega)$, $w_\varepsilon := \phi_\varepsilon(\nabla v_\varepsilon) \in L^2(\Omega)^N$ and $v_\varepsilon - \nabla \cdot w_\varepsilon = g$ in $\mathcal{D}'(\Omega)$.

$\Downarrow \quad \text{letting } \varepsilon \rightarrow 0 \quad \Downarrow$

The proof of the theorem

Numerical approximation : approximation of $\inf_{\sigma \in H_{div}(\Omega)} G(\sigma)$

We consider

- $V : H_{div}(\Omega) \supset V_h$: the space of finite element of Raviard-Thomas
- $r_h : H_{div}(\Omega) \rightarrow V_h$ the projection operator of Raviard-Thomas : for any $w \in H_{div}(\Omega)$, as $h \rightarrow 0$, we have

$$(r_h(w), \operatorname{div}(r_h(w))) \rightarrow (w, \operatorname{div}(w)) \quad \text{in } \left(L^2(\Omega)\right)^{N+1}$$

Theorem (Dumont-Ig,2007)

Let q_h be the solution of

$$G(q_h) = \inf\{G(\sigma_h), \sigma_h \in V_h\}$$

and $q \in M_b(\Omega)^N$ such that $\operatorname{div}(q) \in L^2(\Omega)$ and q is a solution of the solution of

$$G(q) = \inf\{G(\sigma), \sigma \in H_{div}(\Omega)\},$$

then

$$\|\operatorname{div}(q) - \operatorname{div}(q_h)\|_{L^2(\Omega)} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

Related works and works in progress :

- Sandpile problem on non flat regions (on obstacles).
- Nonlocale sandpile problem : stochastic model.
- Moving sand dunes.
- Collapsing sandpile problem.
- Hysterisis : angle of stability is different than the angle of avalanches.

Cas d'une source centrale

Ecoulement d'un tas contre un mur

Ecoulement d'un tas sur une table

Cas d'une source qui tourne

Effondrement d'un tas