Sur le problème d'évolution associé à l'équation de Monge-Kantorovich

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Monge-Kantorovich evolution equation

Let $\Omega \subset \mathbb{R}^N$ be a bounded regular domain, $N \geqslant 1$, $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \partial\Omega \times (0, T)$

$$(EMK) \qquad \begin{cases} \dfrac{\partial h}{\partial t}(x,t) - \nabla \cdot m(x,t)\nabla h(x,t) = \mu & \text{in } Q \\ m(x,t) \geqslant 0, \ |\nabla h(x,t)| \leq 1 & \text{in } Q \\ m\,(1 - |\nabla h(x,t)|) = 0 & \text{in } Q \\ u = 0 & \text{on } \Sigma \\ u(0) = u_0 & \text{in } \Omega \end{cases}$$

Main interest :

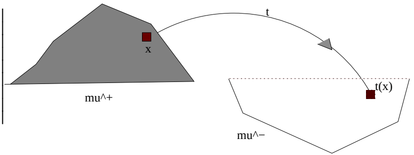
1. Existence and uniqueness of a solution : μ a Radon measure
2. Numerical analysis : representation of $t \in [0, T) \to (u(t,x), m(t,x)\,\nabla u(t,x))$..
3. Large time behavior, as $t \to \infty$.

Motivation I : optimal mass transportation

Monge optimal mass transportation [Monge-1780] : Given two measures μ^+ and μ^- in $\mathbb{R}^N$, such that $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$ and $\text{spt}(\mu^+) \neq \text{spt}(\mu^-)$

$$Min_{t\in\mathcal{A}} \quad \int_{\mathbb{R}^N} |x - t(x)|\, d\mu^+(x)$$

$$\mathcal{A} = \left\{ \begin{array}{l} t \;:\; \text{spt}(\mu^+) \to \text{spt}(\mu^-) \;;\; t\#\mu^+ = \mu^- \\ \text{i.e. } \mu^-(B) = \mu^+(t^{-1}(B)) \end{array} \right\}$$



Monge-Kantorovich problem [Kantorovich-1940] :

- Realaxed variant : optimal plan transport

$$\overline{\mu} \in \Pi(\mu^+, \mu^-) := \left\{ \gamma \in \mathcal{M}_b(\boldsymbol{R}^N \times \boldsymbol{R}^N),\; proj_x(\gamma) = \mu^+,\; proj_y(\gamma) = \mu^- \right\}$$

$$\mathcal{J}(\overline{\mu}) = \min_{\gamma\in\Pi(\mu^+,\mu^-)} \mathcal{J}(\gamma) \qquad\qquad \mathcal{J}(\gamma) = \iint_{R^N\times R^N} |x-y|\, d\gamma(x,y)$$

Is $\overline{\mu}$ supported in a graph ? i.e. $\exists\ ?\ t$, $\overline{\mu}(E) = \mu^+(\{x \in R^N \;;\; (x, t(x)) \in E\})$

- Dual variational principle : Shipper (principe du convoyeur) $\overline{\mu}$ is optimal if and only if

$$\mathcal{J}(\overline{\mu}) = \max_{\xi\in Lip_1(\mathbb{R}^N)} \left\{ \int_{R^N} \xi\, d\mu^+ + \int_{R^N} \xi\, d\mu^- \right\} = \int_{R^N} \overline{u}\, d\mu^+ + \int_{R^N} \overline{u}\, d\mu^- .$$

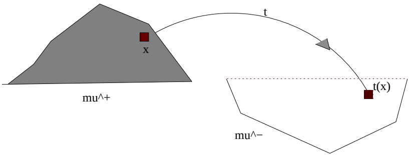
- *Memoire sur la théorie de Déblais et des remblais*, Gaspard Monge, Histoire de l'académie des Sciences de Paris, 1781.
- *On the transfer of massesm*, L.V. Kantorovich, Dokl. Acad. Nauk. SSSR 37, 227-229 (1942).

Motivation I : optimale mass transportation

Monge optimal mass transfer [Monge-1780] : Given two measures μ^+ and μ^- in $\mathbb{R}^N$, such that $\mu^+(\mathbb{R}^N) = \mu^-(\mathbb{R}^N)$ and spt$(\mu^+) \neq$ spt(μ^-)

$$Min_{t\in\mathcal{A}} \quad \int_{\mathbb{R}^N} |x - t(x)| \, d\mu^+(x)$$

$$\mathcal{A} = \left\{ \begin{array}{l} t \ : \ \text{spt}(\mu^+) \to \text{spt}(\mu^-) \ ; \ \mu^+_{\#t} = \mu^- \\ \text{i.e. } \mu^-(B) = \mu^+(t^{-1}(B)) \end{array} \right\}$$



How to fashion an optimal transport t ?

Evans and Gangbo [1999] : Monge-Kantorovich equation (stationnary equation of (EMK))

$$(EG) \qquad \left\{ \begin{array}{l} -\nabla\cdot(m\,\nabla u) = \mu \ \ (:= \mu^+ - \mu^-) \\ m \geqslant 0, \ |\nabla u| \leq 1, \ m(|\nabla u| - 1) = 0 \end{array} \right\} \Longrightarrow m\,\nabla u : \text{the flux transportation}$$

- m gives the transport density
- $-\nabla u$ gives the direction of the optimal transportation

Remark : If (u, m) solves (EG) then u is the Kantorovich potential ; i.e.

$$\max_{\xi\in Lip_1(\mathbb{R}^N)} \int_{R^N} \xi \, d\mu \ = \ \int_{R^N} u \, d\mu$$

Motivation II : Prigozhin model for growing sandpile

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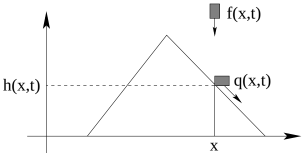


- Conservation of mass :
  - The flow of the granular material is confined in a thin boundary layer moving down the slopes of a growing pile
  - The density of the material is constant

$$\frac{\partial h}{\partial t}(x,t) = -\nabla \cdot q(x,t) + \mu(x,t)$$

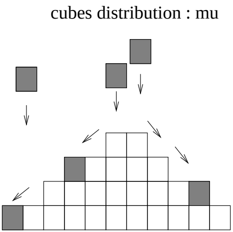
- Surface flow directed by the steepest descent :

$$\exists\ m = m(x,t) \geqslant 0 \text{ such that } q(x,t) = -m(x,t)\,\nabla h(x,t)$$

- No pouring over the parts of the pile surface inclined less than α :

$$|\nabla h(x,t)| < \gamma \Longrightarrow m(x,t) = 0$$

Motivation III : discrete model (stochastic)



The associated continuous model :

N cubes of sides $1/N$, $N \to \infty$ $\Rightarrow$ Evolution Monge-Kantorovich equation

Works in progress :

- Some continuous model associated with the Stochastic approach [Poaty-Ig].
- Stability of protective sea dyke (Projet FLUPARTI soutenue et financé par le conseil regionnal de Picardie)

- *A stochastic model for growing sandpiles and its continuum limit*, L. C. Evans, F. Rezakhanlou, Comm. Math. Phys., 197 (1998), no2, 325-345.

Boundary condition : Dirichlet

- If $\mu^+(\boldsymbol{R}^N) = \mu^-(\boldsymbol{R}^N)$ and μ^+, μ^- are supported in bounded domain : by taking Ω large enough we can assume that $u = 0$ on $\partial\Omega$;

- If $\mu^+(\boldsymbol{R}^N) \neq \mu^-(\boldsymbol{R}^N)$ and μ^+, μ^- are supported in bounded domain :
  - There exists Γ such that $\mathcal{H}^{N-1}(\Gamma) = 0$, one needs to replace $c(x,y) = |x-y|$ by the semi-geodesic distance taking into account the boundary ; i.e.

$$c_\Gamma(x,y) = \min\Big(|x-y|, dist(x,\Gamma) + dist(y,\Gamma)\Big).$$

  - **Dirichlet Monge-Kantorovich problem** (Bouchité, Buttazzo, De Pascale ...)

$$\inf\left\{\int\!\!\int c_\Gamma(x,y)\, d\nu(x,y)\ ;\ \nu \in \mathcal{M}(\Omega\times\Omega) \text{ s.t. } \pi^1_\#\nu - \pi^2_\#\nu = \mu_1 - \mu_2\right\}$$

Références of the talk

http ://www.mathinfo.u-picardie.fr/igbida/

- S. Dumont and N. Igbida,
  Back on a Dual Formulation for the Growing Sandpile Problem, *submitted*.
- N. Igbida,
  Evolution Monge-Kantorovich Equation, *submitted*.
- N. Igbida,
  On Monge-Kantorovich Equation, *Preprint*.

Main Difficulties

$$\begin{cases} \partial_t u - \nabla\cdot(m\,\nabla v) = \mu \text{ in } \mathcal{D}'(\Omega) \\ |\nabla u| \le 1,\ m \geqslant 0,\ m\,(|\nabla u| - 1) = 0 \end{cases} \Rightarrow \begin{cases} u_t + \partial \boldsymbol{I}_K(u) \ni \mu, \\ K = \left\{ z \in H^1_0(\Omega)\ ;\ |z| \le 1 \text{ in } \Omega \right\} \end{cases}$$

i.e. $\forall\ t \in [0, T),\ u(t) \in K$ and $\displaystyle\int_\Omega u(t)\,(\mu(t) - \partial_t u(t)) = \max_{\xi \in K} \int (\mu(t) - \partial_t u(t))\,\xi.$

Nonlinear semi-group theory $\Rightarrow$ existence and uniqueness of "variational" **solution** u

Main Difficulies

- The main information are in the flux $\Phi = m\,\nabla u$:

Monge-Kantorovich	$\longrightarrow$	flux transport
Sandpile	$\longrightarrow$	numerical analysis

- In general, m is not regular : m is a measure.
- How to define the solution for (EG) and (EMK) : m is a measure and $\nabla u \in L^\infty(\Omega)$.
- How to do numerical analysis for (EG) and (EMK) : regularisation technics are not stable.

Notion of solution : how to define $m\,\nabla u$ (m a Radon measure)

Stationnary problem : Theory of tangentiel gradient of u with respect to Radon measure (cf. Bouchitté, Buttazzo, De Pascale, Seppecher ...) :

$$(MK) \qquad \begin{cases} -\nabla\cdot(m\,\nabla_m u) = \mu & \text{in } \mathcal{D}'(\Omega) \\ |\nabla_m u| = 1 & m-\ \text{a.e. in } \Omega. \end{cases} \qquad \text{(Monge-Kantorovich equation)}$$

In particular :

- Let $\nu \in \mathcal{M}_b(\Omega)$ and $1 < p < \infty$.

$$w = \nabla_\nu u \ \in L^p_\nu(\Omega) \Leftrightarrow \exists\ u_h \in \mathcal{C}^1(\Omega),\ (u_h, \nabla u_h) \to (u, w) \quad \text{in } L^p_\nu(\Omega)^{N+1}.$$

- For any $F \in L^{p'}_\nu(\Omega)^N$, such that $-\nabla\cdot(F) = \mu\ \in L^p(\Omega)$, we have

$$\int_\Omega \nabla_\nu u \cdot F\, d\nu = -\int_\Omega u\,\mu$$

for any $u \in \mathcal{C}_0(\Omega)$ such that u is Lipchitz.

Existence, uniqueness of $m\,\nabla_m u$, regularity, related transport for Monge problem... : Ambrosio, Bouchitté, Buttazzo, De Pascale, Evans, Seppecher ....

Notion of solution : how to define $m\,\nabla u$ (m a Radon measure)

Evolution problem :

- There is no theory of tangentiel derivative for time dependent functions $t \in (0,T) \to \nabla_\nu u(t)$.
- For $\mu \in L^2(Q)$: existence, uniqueness and numerical analysis by dual formulation
  - Prigozhin - 1996 : $m \in \left(L^\infty(Q)\right)^*$?

$$\begin{cases} \displaystyle\iint_Q (u_t - \mu)\,\xi + \Big\langle m, \nabla u \cdot \nabla \xi \Big\rangle = 0 \\ \Big\langle m, \xi \Big\rangle \geqslant 0, \quad \Big\langle m, |\nabla u|^2 - 1 >= 0 \end{cases}$$

  - Prigozhin and Barrett - 2006 : $m\,\nabla u =: \Phi \in \mathcal{M}_b(\Omega)^N$, $\nabla \cdot \Phi \in L^2(Q)$.
    - Minimzing problem in $\mathcal{M}_b(\Omega)^N$ related to Φ and $\int_0^t \Phi \implies$ gives Φ
      $u(t) = u_0 + \int_0^t f(s)\,ds + \nabla \cdot \int_0^t \Phi.$
    - This is equivalent

$$\begin{cases} u_t - \nabla \cdot \Phi = \mu \quad \text{in } \mathcal{D}'(Q) \\ \displaystyle\iint_Q u\,\nabla \cdot \Phi = |\Phi|(Q) \end{cases}$$

- **Our aim :** a general theory of existence and uniqueness, large time behavior and numerical analysis.

Notion of solution : a simple characterization of the solution

Remark

For any $A,\ B \in \mathbb{R}^N$ such that $|B| \le 1$ and $m \in \mathbb{R}^+$, the following assertions are equivalent

- $m\,(|B| - 1) = 0$ and $A = m\,B$
- $m = |A| = A \cdot B$

Theorem (Ig,2007)

Let μ be a Radon measure and $u \in K$. Then,

$$\int_\Omega u\,d\mu = \max_{\xi\in K} \int_\Omega \xi\,d\mu \quad \Leftrightarrow \quad \exists \Phi \in \mathcal{M}_b(\Omega)^N \quad \begin{cases} -\nabla\cdot(\Phi) = \mu & in\ \mathcal{D}'(\Omega) \\ |\Phi|(\Omega) \le \displaystyle\int_\Omega u\,d\mu \end{cases} \qquad \textbf{(weak solution)}$$

$$\begin{array}{c}\Rightarrow \\ \Leftarrow\end{array} \quad \begin{cases} -\nabla\cdot(\Phi) = \mu & in\ \mathcal{D}'(\Omega) \\ \Phi = |\Phi|\ \nabla_{|\Phi|} u \end{cases} \quad \begin{array}{c}\Rightarrow \\ \Leftarrow\end{array} \quad \int_\Omega u\,d\mu = |\Phi|(\Omega) = \min\{|\nu|(\Omega)\ ;\ -\nabla\cdot\nu = \mu\}.$$

Remark

The expression $|\Phi|(\Omega) = \displaystyle\int_\Omega u\,d\mu$ is a weak formulation of $|\Phi| = \Phi\cdot\nabla u$.

Equivalence for (MK) : main ideas of the proof

Lemma

Let $u \in W_0^{1,\infty}(\Omega)$ *be s.t.* $|\nabla u| \leq 1$ *a.e. in* Ω. *Then,*

- *for any* $\nu \in \mathcal{M}_b(\Omega)^+$,
$$|\nabla_\nu u| \leq 1 \quad \mu - a.e.\ in\ \Omega.$$
- *there exists* $z_\epsilon \in K \cap \mathcal{D}(\Omega)$ *such that* $z_\epsilon \to z$ *in* $\mathcal{C}_0(\Omega)$.

- If (u, Φ) is a weak solution then $|\Phi|(\Omega) = \int_\Omega u\, d\mu$ and u is a variational solution :
  - $\int_\Omega u\, d\mu = \lim_{\varepsilon \to 0} \int_\Omega \nabla u_\varepsilon \cdot \frac{\Phi}{|\Phi|}\, d|\Phi| \leq |\Phi|(\Omega).$
  - $\int_\Omega \xi\, d\mu = \int_\Omega \frac{\Phi}{|\Phi|} \cdot \nabla \xi\, d|\Phi| \leq |\Phi|(\Omega) = \int_\Omega u\, d\mu.$
- If (u, Φ) is a weak solution then (u, Φ) is a solution of (MK) :
$$\begin{cases} |\Phi|(\Omega) = \int_\Omega u\, d\mu = \int_\Omega \nabla_{|\Phi|} u \cdot \frac{\Phi}{|\Phi|}\, d|\Phi| \\ |\nabla_{|\Phi|} u| \leq 1 \quad |\Phi|\text{-a.e. in } \Omega \end{cases} \Longrightarrow \quad \nabla_{|\Phi|} u \cdot \frac{\Phi}{|\Phi|} = 1 \quad |\Phi| - \text{a.e. in } \Omega.$$
- If (u, Φ) satisfies (MK), then (u, Φ) is a weak solution
$$|\Phi|(\Omega) = \int_\Omega |\nabla_{|\Phi|} u|^2\, d|\Phi| = \int_\Omega \frac{\Phi}{|\Phi|} \cdot \nabla_{|\Phi|} u\, d|\Phi| = \int_\Omega u\, d\mu$$
- If u is a variational solution, then $\exists \Phi$ such that (u, Φ) is a weak solution : $p \to \infty$ in $\Delta_p u_p = \mu$.

Notion of solution : Evolution Monge-Kantorovich equation

Theorem (Ig,2007)

For any $\mu \in L^1(0,T; w^* - \mathcal{M}_b(\Omega))$ *and* $u_0 \in K$, *there exists* (u,Φ) *a weak solution of (EMK) i.e.* $u \in \mathcal{C}([0,T); L^2(\Omega))$, $u(t) \in K$ *for any* $t \in [0,T)$, $u(0) = u_0$, $\Phi \in \mathcal{M}_b(Q)^n$, $\partial_t u - \nabla \cdot \Phi = \mu$ *in* $\mathcal{D}'(Q)$ *and*

$$\sup_{\eta \in \mathcal{C}_0(\Omega)^N,\ \|\eta\|_\infty \leq 1} \int\int_Q \sigma \frac{\Phi}{|\Phi|} \cdot \eta \, d|\Phi| = \frac{1}{2}\int_0^T \int_\Omega u^2 \sigma_t + \int_0^T \int_\Omega u\, \sigma \, d\mu \qquad (1)$$

for any $\sigma \in \mathcal{D}(0,T)$. *Moreover,* u *is the unique variational solution.*

Remark

- Formally, (u,Φ) satifies

$$\begin{cases} \partial_t u - \nabla \cdot \Phi = \mu, \quad |\nabla u| \leq 1 & \text{in} \quad Q \\ |\Phi(t)|(\Omega) = \displaystyle\int_\Omega u \, d\mu - \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 & \text{in} \quad (0,T). \end{cases}$$

- It is not clear if (1) implies that

$$|\Phi(t)|(\Omega) = \int_\Omega u \, d\mu - \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 \quad \text{in} \quad \mathcal{D}'(0,T).$$

Numerical approximation

$$\text{Assume that } \mu = f \quad \in L^2(0,T;L^2(\Omega)).$$

Time discretisation : Euler implicit schema.

- $0 < t_0 < t_1 < ... < t_n = T \qquad$ whith $t_i - t_{i-1} = \varepsilon \quad \forall i = 1, ..., n$
- $f_1, ..., f_n \in L^2(\Omega))$ such that

$$\sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_{L^2(\Omega)} \leq \varepsilon.$$

- consider the approximation of $u(t)$ by :

$$u_\varepsilon(t) = \begin{cases} u_0 & \text{if } t \in]0, t_1] \\ u_i & \text{if } t \in]t_{i-1}, t_i] \qquad i = 1, ..., n \end{cases}$$

  where u_i is given by :

$$u_i + \partial \mathbf{I}_K(u_i) = \varepsilon f_i + u_{i-1}$$

  for $i \geqslant 1$ whith $u_{i=0} = u_0$.

- u_ε is the $\varepsilon-$approximate solution of (P_μ).

Numerical approximation : convergence of the $\varepsilon-$approximate solution

Nonlinear semigroup theory (in $L^2(\Omega)$)

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

Theorem

Let $u_0 \in L^2(\Omega)$ be such that $|\nabla u_0| \leq 1$ and $f \in BV(0,T;L^2(\Omega)$. Then,

1. *For any $\varepsilon > 0$ and any $\varepsilon-$discretisation, there exists a unique $\varepsilon-$approximation u_ε of (P_f).*
2. *There exists a unique $u \in \mathcal{C}([0,T);L^2(\Omega))$ such that $u(0) = u_0$, and*
$$\|u - u_\varepsilon\|_{C([0,T);L^2(\Omega))} \leq C(\varepsilon) \quad \to 0 \text{ as } \varepsilon \to 0.$$
3. *The function u given by 2. is the unique variational solution of (P_μ).*

Numerical approximation : a generic problem

The generic problem is

$$v + \partial \boldsymbol{I}_K(v) = g$$

$$\Downarrow\Uparrow \qquad \Downarrow\Uparrow \qquad \Downarrow\Uparrow \qquad \Downarrow\Uparrow$$

$$v = \boldsymbol{P}_K g$$

where

- $K = \Big\{ z \in W^{1,\infty}(\Omega) \cap W_0^{1,2}(\Omega) \ ; \ |\nabla u| \leq 1 \Big\}.$
- $\boldsymbol{P}_K$ the projection onto the convex K, with respect to the $L^2(\Omega)$ norm :

$$v = \mathbf{P}_K(u) \Leftrightarrow v \in K, \ \int_\Omega (v-u)(v-z) \geqslant 0 \quad \text{for any } z \in K.$$

Question :

For a given $g \in L^2$, how to compute $v = \boldsymbol{P}_K g$?

Numerical approximation : Dual and Primal formulation

- We define
$$J(z) = \frac{1}{2}\int_\Omega |z-g|^2$$
.
- We denote by
$$H_{div}(\Omega) = \{\sigma \in (L^2(\Omega))^N \ ; \ \mathrm{div}(\sigma) \in L^2(\Omega)\}$$

 and
$$G(\sigma) = \frac{1}{2}\int_\Omega (\mathrm{div}(\sigma))^2 + \int_\Omega g\mathrm{div}(\sigma) + d\int_\Omega |\sigma|.$$

Lemma

$$sup_{\sigma\in H_{div}(\Omega)}\Big(-G(\sigma)\Big) \leq \min_{z\in K} J(z)$$

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Theorem (Dumont-Ig,2007)

Let $g \in L^2(\Omega)$ and $v = \boldsymbol{P}_k(g)$. Then, there exists a sequence $(w_p)_{p\in N}$ in $H_{div}(\Omega)$, such that, as $p \to \infty$,

- $\int_\Omega |w_p| \to \int_\Omega v\,(g-v)$
- $div(w_p) \to v-g \qquad in\ L^2(\Omega)$
- $\lim_{p\to\infty}\Big(-G(w_p)\Big) = \sup_{w\in H_{div}(\Omega)}\Big(-G(w)\Big) = \min_{z\in L^2(\Omega)} J(z) = \frac{1}{2}\int_\Omega |g-v|^2.$

Numerical approximation : main idea of the proof

We consider the following elliptic equation

$$(S_\varepsilon) \qquad \begin{cases} v_\varepsilon - \nabla \cdot w_\varepsilon = g \\ w_\varepsilon = \phi_\epsilon(\nabla v_\varepsilon) \end{cases} \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega.$$

where, for any $\varepsilon > 0$, $\phi_\varepsilon \ : \ \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\phi_\varepsilon(r) = \frac{1}{\varepsilon}(|r| - 1)^+ \frac{r}{|r|}, \qquad \text{for any } r \in \mathbb{R}^N.$$

The nonlinearity ϕ_ε satisfies

i) for any $r_1, \ r_2 \ \in \mathbb{R}^N$, $(\phi_\varepsilon(r_1) - \phi_\varepsilon(r_2)) \cdot (r_1 - r_2) \geqslant 0$

ii) there exists $\varepsilon_0 > 0$ and $A > 1$ such that $\phi_\varepsilon(r) \cdot r \geqslant |r|^2$ for any $r \geqslant A$ et $\varepsilon < \varepsilon_0$.

iii) for any $\varepsilon > 0$ and $r \in \mathbb{R}$, $|\phi_\varepsilon(r)| \leq \phi_\varepsilon(r) \cdot r$

$$\Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow \quad \Downarrow$$

For any $g \in L^2(\Omega)$, (S_ε) has a unique solution v_ε, in the sense that $v_\varepsilon \in H_0^1(\Omega)$, $w_\varepsilon := \phi_\varepsilon(\nabla v_\varepsilon) \in L^2(\Omega)^N$ and $v_\varepsilon - \nabla \cdot w_\varepsilon = g$ in $\mathcal{D}'(\Omega)$.

$$\Downarrow \qquad \text{letting } \varepsilon \to 0 \qquad \Downarrow$$

The proof of the theorem

Numerical approximation : approximation of $\inf_{\sigma \in H_{div}(\Omega)} G(\sigma)$

We consider

- $V : H_{div}(\Omega) \supset V_h$: the space of finite elment of Raviard-Thomas
- $r_h : H_{div}(\Omega) \to V_h$ the projection operator of Raviard-Thomas : for any $w \in H_{div}(\Omega)$, as $h \to 0$, we have

$$(r_h(w), \mathrm{div}(r_h(w))) \to (w, \mathrm{div}(w)) \quad \text{in } \left(L^2(\Omega)\right)^{N+1}$$

Theorem (Dumont-Ig,2007)

Let q_h be the solution of

$$G(q_h) = \inf\{G(\sigma_h),\ \sigma_h \in V_h\}$$

and $q \in M_b(\Omega)^N$ such that $div(q) \in L^2(\Omega)$ and q is a solution of the solution of

$$G(q) = \inf\{G(\sigma),\ \sigma \in H_{div}(\Omega)\},$$

then

$$\|div(q) - div(q_h)\|_{L^2(\Omega)} \to 0 \quad \text{as } h \longrightarrow 0.$$

Related works and works in progress :

- Sandpile problem on non flat regions (on obstacles).
- Nonlocale sandpile problem : stochastic model.
- Moving sand dunes.
- Collapsing sandpile problem.
- Hysterisis : angle of stability is different than the angle of avalanches.

Cas d'une source centrale

Ecoulement d’un tas contre un mur

Ecoulement d'un tas sur une table

Cas d'une source qui tourne

Effondrement d'un tas