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Relation between roots and coefficients, interpolation and application to system solving

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Abstract

We proposed here an algebraic framework to represent zero-dimensional complete intersection algebraic systems (n equations in n variables). In this framework, we give new interpolation formulae. We use this good representation of the algebraic systems to develop a generalization of the Weierstrass's method to the multivariate systems. This method allows to approximate simultaneously all the roots of an algebraic system. We obtain an effective iteration function with a quadratic convergence in a neighborhood of the solutions. We use this Weierstrass's iteration function in a continuation method to obtain a global method. Experiments are exposed to underline the efficiency of the approach.

1. Introduction

The relations between roots and coefficients of an univariate polynomial, involving the elementary symmetric functions, are well known. Exploiting their properties yields efficient methods to approximate the roots of the polynomial, which consist in applying Newton's method to those relations. These methods are known under the names of Weierstrass's method (33), Durand-Kerner's method (21), Aberth's method (1), (7), (34), (5).

The aim of this paper is first to describe such relations, in the multivariate setting, secondly to exploit them for solving polynomial systems. We will consider in a first part the correlation between roots and the structure of the quotient algebra associated with the roots. A special attention will be given to the properties of the dual space, which are fundamental in this context. In the next section, we will consider the multivariate interpolation problem of computing polynomials with fixed value at fixed points and the multivariate Hermite interpolation problem of computing polynomials with fixed tangential conditions at given points. See (22; 24; 11) for more background. We give explicit formulae, for the solutions of such problems. These results are exploited to deduce relations which generalize to the multivariate case, the well-known relations between roots and coefficients of univariate polynomials. They are described in the general context of roots with multiplicity.

This allows us to tackle the solving problem, which consists to find the roots when we know the coefficients. Indeed we consider the inverse problem which consists, just as in the univariate case, to adapt the roots so that the coefficients match those of the given polynomial. For this purpose, we apply Newton iteration to this system of relations between roots and coefficients. The previous analysis yields a new and explicit formula for the inverse of the Jacobian matrix, which generalizes the Weierstrass's formula to the multivariate case. Combining this explicit Newton iteration step with a homotopy method, we obtain a new global method for solving zero-dimensional complete intersection systems, based on the knowledge of a basis of the quotient algebra. Our approach follows the general frame of algebraic solvers, which usually proceed in two distinct steps:

1. Extract informations about the structure of the quotient algebra associated with the algebraic system. For instance, the structure of the vector space and some multiplication operators.
2. Solve a simpler problem. For instance an eigenvalue problem obtained thanks to the knowledge of the quotient's vector space structure and multiplication operator.

There are several efficient algorithms to perform step 1, for instance (16) and (27). We will assume that a preprocessing step (based for instance on modular computations) will give us a basis of the quotient algebra. But the computations for step 2 usually remain heavy.

For the numerical part, we use tools from homotopy theory for polynomial systems to develop a continuation method to compute simultaneously all the solutions. This method is connected with the continuation methods exposed in (12), (30), (20), (23) and (32). For these methods, one path gives one solution. With our algorithm, one path gives all the solutions.

An advantage of this approach is that for solving a set of polynomial systems of the same "shape", for which some of the parameters are changed, we do not need to change the basis. Another advantage of the approach is that only few informations on the vector space structure (a monomial basis) are needed. We do not need to compute normal forms or multiplication operators here. So we can compute a basis using modular arithmetic for instance. Consequently, the step of preprocessing is simplified.

Part of this work as already been presented in (29). In this paper, we add a new description of the quotient structure in the case of multiple roots, a new and explicit formula for the general Hermite interpolation problem and the description between roots and coefficients in the case of multiple roots. More complete experimentations in the C++ environment ALP* for symbolic and numeric computations are also illustrating the potential power of this approach.

Hereafter $\delta_{i,j}$ denotes the Kronecker symbol (it equals 1 if $i = j$ and 0 otherwise). We denote by $\mathbb{K}$ a field of characteristic 0 and by $R = \mathbb{K}[x_1, \ldots, x_n]$ the ring of polynomials in the variables $x_1, \ldots, x_n$, with coefficients in $\mathbb{K}$. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $\mathbf{x}^\alpha$ is the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The sub-vector space generated by the elements $v_1, \ldots, v_k$ of a $\mathbb{K}$-vector space is denoted by $< v_1, \ldots, v_k >$.

2. Univariate case

Let us recall, in this section, the so-called Weierstrass's method. It is a numerical and iterative method, which allows to approximate simultaneously all (or several) roots of an univariate polynomial. Nowadays, there are several methods that generalize this approach in the univariate case (see (28), (5) and (7), for a good surveys). Here, we do not describe the general case, but we focus on the case of a polynomial with only simple complex roots, in order to illustrate what will happen with several variables. This approach consists in finding a system of multivariate equation which admits as "only solution" (up to permutation) a vector formed by the roots of the initial equation and to apply the Newton's method to this new system. Several simplifications occur due to the good knowledge of the quotient algebra associated with the initial polynomial as we will see now.

Let $f \in \mathbb{K}[x]$ be a polynomial of the form

$$f(x) = x^d + a_1 x^{d-1} + \cdots + a_d = \prod_{i=1}^{d} (x - \zeta_i).$$

We denote by $\zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{K}^d$. We will construct a multivariate system of equations which admits as set of solutions, vectors formed with the coordinates of ζ (i.e. the roots of f).

Let $\mathbf{z} \in \mathbb{K}^d$, $\mathbf{z} = (z_1, \ldots, z_d)$, we denote by $\sigma_k(\mathbf{z}) = (-1)^k \sum_{0 \le i_1 < \cdots < i_k \le d} z_{i_1} \cdots z_{i_k}$ the k-th elementary symmetric function. By expanding the expression $\prod_{i=1}^{d} (x - \zeta_i)$, we obtain the following identity : $\sigma_k(\zeta) = a_k, \forall k \in \{1, \ldots, d\}$. Let us define the following application :

$$\Sigma(\mathbf{z}) = \begin{pmatrix} \sigma_1(\mathbf{z}) - a_1 \\ \vdots \\ \sigma_d(\mathbf{z}) - a_d \end{pmatrix} \tag{1}$$

*http://www-sop.inria.fr/galaad/logiciels/ALP/

This system defines the well known relations between roots and coefficients. We denote by $\mathcal{S}_n$ the set of permutation of the set $\{1,\ldots,d\}$. Then we easily verify that $\mathbf{z}\in\mathbb{K}^d$ is such that $\Sigma(\mathbf{z})=0$, if and only if there exists a permutation $\tau\in\mathcal{S}_n$ such that $\mathbf{z}^\tau=\left(z_{\tau(1)},\ldots,z_{\tau(d)}\right)=\zeta$.

Let $\mathbf{z}\in\mathbb{K}^n$, we define the polynomials $f_{\mathbf{z}}(x)=\sum_{i=1}^{d}(x-z_i)$ and $F_{\mathbf{z}}(x)=f_{\mathbf{z}}(x)-f(x)$. We denote by $\mathbb{K}[x]_{d-1}$ the set of polynomials of degree less or equal to $d-1$, $\mathcal{A}_{\mathbf{z}}=\mathbb{K}[x]/(f_{\mathbf{z}})$ and $\mathcal{A}=\mathcal{A}_\zeta=\mathbb{K}[x]/(f)$, which are vector spaces of dimension d. Another way to look at the roots of f is to consider the zero of the following application:

$$\overrightarrow{F}:\left\{\begin{array}{ccc}\mathbb{K}^d & \longrightarrow & \mathbb{K}[x]_{d-1}\\ \mathbf{z} & \longmapsto & F_{\mathbf{z}}(x)\end{array}\right.$$

Indeed, we check that $\overrightarrow{F}(\mathbf{z})=0$ if and only if $\Sigma(\mathbf{z})=0$. Let us introduce the Lagrange's polynomials associated with the roots z_i:

$$\mathbf{e}_{z_i}=\frac{\prod_{j\neq i}(x-z_j)}{\prod_{j\neq i}(z_i-z_j)}$$

which yield the decomposition $\mathcal{A}_{\mathbf{z}}=\mathbb{K}\mathbf{e}_{z_1}\oplus\cdots\oplus\mathbb{K}\mathbf{e}_{z_d}$.

An idea to approximate simultaneously all the roots of f is to apply Newton's method to $\Sigma(\mathbf{z})$ or to $\overrightarrow{F}$. For that, we need to compute

$$Jac_{\overrightarrow{F}}(\mathbf{z})^{-1}\,\overrightarrow{F}(\mathbf{z})=\mathbf{u},$$

or equivalently to solve the linear system

$$Jac_{\overrightarrow{F}}(\mathbf{z})\,\mathbf{u}=\overrightarrow{F}(\mathbf{z}).\tag{2}$$

or in other words

$$\sum_{i=1}^{d}\frac{\partial}{\partial z_i}(F_{\mathbf{z}})\,u_i=F_{\mathbf{z}}.$$

Since we have $\frac{\partial}{\partial z_i}F_{\mathbf{z}}=\frac{\partial}{\partial z_i}\prod_{j=1}^{d}(x-z_j)=-\prod_{j\neq i}(x-z_j)=-\left(\prod_{j\neq i}(z_i-z_j)\right)\mathbf{e}_{z_i}$ and since

$$F_{\mathbf{z}}(x)=\sum_{i=1}^{d}F_{\mathbf{z}}(z_i)\,\mathbf{e}_{z_i}=\sum_{i=1}^{d}-f(z_i)\,\mathbf{e}_{z_i},$$

we deduce, by identification, that we have :

$$u_i=\frac{f(z_i)}{\prod_{j\neq i}(z_i-z_j)}.\tag{3}$$

Thanks to this formula, we have a very simple way to express recursively the sequence $\left(\mathbf{z}^{(k)}\right)_{k\in\mathbb{N}}$ of the points of the Newton's method :

$$\forall i \in \{1, \ldots, d\}, \; z_i^{(k+1)} = \mathbf{I}_i^{DK}(\mathbf{z}) = z_i^{(k)} - \frac{f\left(z_i^{(k)}\right)}{\prod_{j\neq i}\left(z_i^{(k)} - z_j^{(k)}\right)} \tag{4}$$

It allows us to describe the coordinates of the iteration function of the so-called Durand-Kerner's method :

$$\mathbf{z}^{(k+1)} = \mathbf{I}^{DK}\left(\mathbf{z}^{k}\right) = \left(\mathbf{I}_1^{DK}\left(\mathbf{z}^{(k)}\right), \ldots, \mathbf{I}_d^{DK}\left(\mathbf{z}^{(k)}\right)\right) \tag{5}$$

avoiding the inversion of the Jacobian matrix in Newton's iteration. The key ingredients are the Lagrange interpolation basis and its dual basis. The multivariate case, though more intricate, follows the same lines as we will see.

3. Algebraic structure

In this section, we introduce the algebraic tools that we will use hereafter. We recall basics on algebraic sets and ideal decomposition before giving explicit algorithms and formulae for interpolation and relation between roots and coefficients. Let $f_1, \ldots, f_n$ be polynomials of $\mathbb{K}[x_1, \ldots, x_n]$. We denote by $\mathcal{I} = (f_1, \ldots, f_n)$ the ideal that they generate and by $Z(\mathcal{I}) = \left\{\mathbf{z} \in \overline{\mathbb{K}}^n \mid g(\mathbf{z}) = 0, \, \forall g \in \mathbf{f}\right\} = \{\mathbf{z} \in \mathbb{K}^n \mid f_i(\mathbf{z}) = 0, \forall i \in \{1, \ldots, n\}\}$ the algebraic set they define. We denote by $\mathcal{A}$ the quotient algebra $\mathbb{K}[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.

HYPOTHESIS 3.1: *We assume that $Z(f_1, \ldots, f_m)$ is a finite set $\{\zeta_1, \ldots, \zeta_d\}$.*

Under the hypothesis 3.1, the ideal $\mathcal{I}$ is 0-dimensional and the minimal primary decomposition of $\mathcal{I}$ is $\mathcal{I} = Q_{\zeta_1} \cap \cdots \cap Q_{\zeta_d}$ where Q_{ζ_i} is $\mathbf{m}_{\zeta_i}$-primary and $\mathbf{m}_{\zeta_i}$ is the maximal ideal defining ζ_i (31).

3.1. Decomposition of the quotient algebra

We denote by $\mathcal{A}_{\zeta_i} = ((\mathbf{0}) : Q_{\zeta_i}/\mathcal{I})$ (i.e. $\mathcal{A}_{\zeta_i} = \{a \in \mathcal{A} | \forall q \in Q_{\zeta_i}/\mathcal{I}, \, ap = 0\}$). It is an ideal of $\mathcal{A}$. Because Q_{ζ_i} is a primary ideal for the maximal ideal $\mathbf{m}_{\zeta_i}$ we have the following proposition (3), (31), (14):

PROPOSITION 3.2: *The algebra $\mathcal{A}$ is the direct sum of its sub-algebra $\mathcal{A}_{\zeta_1}, \ldots, \mathcal{A}_{\zeta_d}$, i.e. $\mathcal{A} = \mathcal{A}_{\zeta_1} \oplus \cdots \oplus \mathcal{A}_{\zeta_d}$.*

DEFINITION 3.3: *The multiplicity of the root $\zeta_i \in Z(\mathcal{I})$ is the dimension of the sub-vector space $\mathcal{A}_{\zeta_i}$ of $\mathcal{A}$. This dimension is denote μ_i. The root is simple if $\mu_i = 1$ and multiple if $\mu_i > 1$.*

From this definition and the previous proposition we deduce that the dimension of $\mathcal{A}$ as a $\mathbb{K}$-vector space is the number of root (i.e. element of $Z(\mathcal{I})$) counted with theirs multiplicities. That is to say that $\dim_{\mathbb{K}}(\mathcal{A}) = \mu_{\zeta_1} + \cdots + \mu_{\zeta_d}$.

There exists an unique d-uple $(\mathbf{e}_{\zeta_1}, \ldots, \mathbf{e}_{\zeta_d}) \in \mathcal{A}_{\zeta_1} \times \cdots \times \mathcal{A}_{\zeta_d}$ such that $1 = \mathbf{e}_{\zeta_1} + \cdots + \mathbf{e}_{\zeta_d}$. Then we have $1 = \mathbf{e}_{\zeta_1} + \cdots + \mathbf{e}_{\zeta_d} = 1^2 = \mathbf{e}_{\zeta_1}^2 + \cdots + \mathbf{e}_{\zeta_d}^2 + 2 \sum_{1 \leq i < j \leq d} \mathbf{e}_{\zeta_i} \mathbf{e}_{\zeta_j}$. Because $\mathbf{e}_{\zeta_i} \mathbf{e}_{\zeta_j} \in \mathcal{A}_{\zeta_i} \cap \mathcal{A}_{\zeta_j} = (\mathbf{0})$ if $i \neq j$, then we deduce that $\mathbf{e}_{\zeta_i} \mathbf{e}_{\zeta_j} = 0$ and $\mathbf{e}_{\zeta_i}^2 = \mathbf{e}_{\zeta_i}$. The polynomials $\mathbf{e}_{\zeta_i}$ are called the fundamental system of idempotents of $\mathcal{A}$. As we will see, they generalize the Lagrange interpolation polynomials.

Proposition 3.4: *For $i \in \{1, \ldots, d\}$, we have $\mathcal{A}_{\zeta_i} = \mathcal{A}\mathbf{e}_{\zeta_i}$ and so $\mathbf{e}_{\zeta_i}$ is the neutral element of the sub-algebra $\mathcal{A}_{\zeta_i}$.*

Proof: If $a \in \mathcal{A}_{\zeta_i}$, then $a = a1 = a\mathbf{e}_{\zeta_1} + \cdots + a\mathbf{e}_{\zeta_d} = a\mathbf{e}_{\zeta_i}$. The reciprocal is easy. □

Proposition 3.5: *The idempotents $\mathbf{e}_{\zeta_1}, \ldots, \mathbf{e}_{\zeta_d}$ of $\mathcal{A}$ are such that $\mathbf{e}_{\zeta_i}(\zeta_j) = \delta_{i,j}$.*

Proof: For $i, j \in \{1, \ldots, d\}$ and $j \neq i$, there exists $q \in Q_{\zeta_i}$ such that $q(\zeta_j) \neq 0$. Indeed, if there is no such polynomial than $Q_{\zeta_i} \subset \mathit{displaystyle} \cup_{j \neq i} \mathbf{m}_{\zeta_i}$ and Q_{ζ_i} must be include in one of the $\mathbf{m}_{\zeta_j}$ with $j \neq i$. It yields to a contradiction since Q_{ζ_i} is $\mathbf{m}_{\zeta_i}$-primary. Now, because $q\mathbf{e}_{\zeta_i} = 0$, $\mathbf{e}_{\zeta_i}(\zeta_j) = 0$ and i follows that $1 = \mathbf{e}_{\zeta_1}(\zeta_i) + \cdots + \mathbf{e}_{\zeta_d}(\zeta_i) = \mathbf{e}_{\zeta_i}(\zeta_i)$. □

We know that if $\mu_i > 1$, then $\mathcal{A}_{\zeta_i}$ is a $\mathbb{K}$-vector space of dimension $\mu_i > 1$ and $\mathcal{A}_{\zeta_i} = \mathcal{A}\mathbf{e}_{\zeta_i}$ is a $\mathcal{A}$-module of rank one. To have a better understanding of the structure we will first consider the projection map of $\mathcal{A}$ on $\mathcal{A}_{\zeta_i}$.

Consider the following onto $\mathbb{K}$-linear map:

$$\pi_{\zeta_i} : \left\{ \begin{array}{ccc} \mathcal{A} & \longrightarrow & \mathcal{A}_{\zeta_i} \\ a & \longmapsto & a\mathbf{e}_{\zeta_i} \end{array} \right.$$

Proposition 3.6: *$Ker(\pi_{\zeta_i}) = \oplus_{j \neq i} \mathcal{A}_{\zeta_j} = Q_{\zeta_i}/\mathcal{I}$.*

Proof: $\oplus_{j \neq i} \mathcal{A}_{\zeta_j} = \left((0) : \cap_{j \neq i} Q_{\zeta_j}/\mathcal{I}\right) = Q_{\zeta_i}/\mathcal{I}$. □

Henceforth, we identify $\mathcal{A}_{\zeta_i}$ with $\mathcal{A}/\mathrm{Ker}(\pi_{\zeta_i})$. But $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$ and $\mathrm{Ker}(\pi_{\zeta_i}) = Q_{\zeta_i}/\mathcal{I}$ and so $\mathcal{A}_{\zeta_i} = \mathbb{K}[x_1, \ldots, x_n]/Q_{\zeta_i}$. This expression allows us to have a good representation of $\mathcal{A}_{\zeta_i}$ thanks to the following proposition:

Proposition 3.7: *The map $\Phi_{\zeta_i} : \left\{ \begin{array}{ccc} \mathbb{K}[x_1, \ldots, x_n]/Q_{\zeta_i} & \longrightarrow & \mathcal{A}_{\zeta_i} \\ a & \longmapsto & a\mathbf{e}_{\zeta_i} \end{array} \right.$ is an isomorphism of $\mathbb{K}$-algebras.*

Thanks to the fact that Q_{ζ_i} is $\mathbf{m}_{\zeta_i}$-primary and to the proposition 3.7 we have a lot of information in the structure of $\mathcal{A}_{\zeta_i}$.

Proposition 3.8: *The algebra $\mathcal{A}_{\zeta_i}$ is a local ring of maximal ideal $(\mathbf{m}_{\zeta_i}/Q_{\zeta_i})\,\mathbf{e}_{\zeta_i}$.*

To describe completely the structure of $\mathcal{A}_{\zeta_i}$ as a $\mathbb{K}$-vector space we introduce hereafter some tools of duality.

3.2. Duality and orthogonality

DEFINITION 3.9: *We denote by* $\widehat{\mathcal{R}} = Hom_{\mathbb{K}}(\mathcal{R}, \mathbb{K})$*, the dual space of* $\mathcal{R}$*. It is the set of* $\mathbb{K}$*-linear forms on* $\mathcal{R}$*.*

We recall the interpretation of the $\mathbb{K}$-linear forms as differential operators (we refer to (25)). We denote by ∂_i the operator $\frac{d}{dx_i}$ and for all $\alpha \mathbb{N}^n$ we denote by $\partial^\alpha = \frac{d^{\alpha_1}}{dx_1{}^{\alpha_1}} \cdots \frac{d^{\alpha_n}}{dx_n{}^{\alpha_n}}$ with the convention $\partial^{\mathbf{0}} = 1$. We denote by $\mathbb{K}[[\partial_1, \ldots, \partial_n]]$ the set of formal power series in the variables $\partial_1, \ldots, \partial_n$. Let $\xi \in \mathbb{K}^n$, we consider the following bilinear map:

$$< .,. >_\xi : \left\{ \begin{array}{ccc} \mathbb{K}[[\partial_1, \ldots, \partial_n]] \times \mathcal{R} & \longrightarrow & \mathbb{K} \\ (\Lambda, f) & \longmapsto & < \Lambda, f >_\xi = \Lambda(f)(\xi) \end{array} \right.$$

where if $\Lambda = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha \partial^\alpha$, then $\Lambda(f)(\xi) = \sum_{\alpha \in \mathbb{N}^n} \lambda_\alpha (\partial^\alpha f)(\xi)$.

The map $\Theta_\xi : \left\{ \begin{array}{ccc} \mathbb{K}[[\partial_1, \ldots, \partial_n]] & \longrightarrow & \widehat{\mathcal{R}} \\ \Lambda & \longrightarrow & \Lambda \circ 1_\xi = < \Lambda, . >_\xi \end{array} \right.$ is an isomorphism of $\mathbb{K}$-vector spaces which allows us to identify $\widehat{\mathcal{R}}$ with $\mathbb{K}[[\partial_1, \ldots, \partial_n]]$.

Let $f \in \mathcal{R}$ and $\Lambda \in \widehat{\mathcal{R}}$ we define $f\Lambda : g \in \mathcal{R} \mapsto \Lambda(fg) \in \mathbb{K}$, so that $\widehat{\mathcal{R}}$ has a natural structure of $\mathcal{R}$-module. We can show that the multiplication by $x_i - \xi_i$ act on $\widehat{\mathcal{R}}$ as ∂_{∂_i}.

One can remark that, if $\mathbb{K}$ has the characteristic zero, $< \partial^\beta, (\mathbf{x} - \xi)^\alpha >_\xi = \delta_{\alpha,\beta} \alpha!$ and so that $\left(\frac{1}{\alpha!}\partial^\alpha\right)_{\alpha \in \mathbb{N}^n}$ and $((\mathbf{x} - \xi)^\alpha)_{\alpha \in \mathbb{N}^n}$ are dual basis for the bilinear form $< .,. >_\xi$.

We denote by $e^{(\xi,\partial)} = \sum_{\alpha \in N^n} \frac{\xi^\alpha \partial^\alpha}{\alpha!}$. Let $\Lambda \in \widehat{\mathcal{R}}$ be a linear form. One can show that $\Theta_{\mathbf{0}}^{-1}(\Lambda) = e^{(\xi,\partial)} \Theta_\xi^{-1}(\Lambda)$.

We will now define a notion of orthogonality:

DEFINITION 3.10:
- *Let* $\mathcal{I} \subset \mathcal{R}$ *be an ideal, we define the sub-vector space* $\mathcal{I}^\perp \subset \widehat{\mathcal{R}}$ *orthogonal to* $\mathcal{I}$ *(also called the inverse system of* $\mathcal{I}$*) by:*

$$\mathcal{I}^\perp = \left\{ \Lambda \in \widehat{\mathcal{R}} \,|\, \Lambda(f) = 0, \forall f \in \mathcal{I} \right\}.$$

- *Let* D *be a sub-vector space of* $\widehat{\mathcal{R}}$*, we define the sub-vector space* $D^\perp$ *of* $\mathcal{R}$ *orthogonal to* D *by:*

$$D^\perp = \{ f \in \mathcal{R} \,|\, \Lambda(f) = 0, \forall \Lambda \in D \}.$$

PROPOSITION 3.11: *Let* $\mathcal{I}$ *be an ideal of* $\mathcal{R}$ *and* $\mathcal{A} = \mathcal{R}/\mathcal{I}$*, then the projection* $\pi : \mathcal{R} \to \mathcal{A}$ *induce an isomorphism* $\pi_* : \widehat{\mathcal{A}} \to \mathcal{I}^\perp$*.*

THEOREM 3.12: *There is a one to one correspondence between ideals of* $\mathcal{R}$ *and sub-vector spaces of* $\widehat{\mathcal{R}}$ *stable by derivation and closed for the* $(\partial_1, \ldots, \partial_n)$*-adic topology.*

Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and D a closed sub-vector space of dR stable by derivation, we have:

$$\mathcal{I}^{\perp^{\perp}} = \mathcal{I} \text{ and } D^{\perp^{\perp}} = D.$$

DEFINITION 3.13: *Let $\Lambda_1, \ldots, \Lambda_s \in \mathbb{K}[[\partial_1, \ldots, \partial_n]]$, we denote by $<< \Lambda_1, \ldots, \Lambda_s >>$ the sub-vector space of $\mathbb{K}[[\partial_1, \ldots, \partial_n]]$ generated by $\Lambda_1, \ldots, \Lambda_s$ and their derivatives.*

We refer to (25) for a more complete description of this formalism. We just recall that if $\mathcal{I}$ and $\mathcal{J}$ are two ideals of $\mathcal{R}$ then $\mathcal{I}^{\perp^{\perp}} = \mathcal{I}$, $\mathcal{I} \subset \mathcal{J}$ if and only if $\mathcal{J}^{\perp} \subset \mathcal{I}^{\perp}$, $(\mathcal{I} \cap \mathcal{J})^{\perp} = \mathcal{I}^{\perp} + \mathcal{J}^{\perp}$ and $(\mathcal{I} + \mathcal{J})^{\perp} = \mathcal{I}^{\perp} \cap \mathcal{J}^{\perp}$. The properties of orthogonality with respect to ideal division give the following proposition.

PROPOSITION 3.14: *Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ such that $Z(\mathcal{I}) = \{\zeta_1, \ldots, \zeta_d\}$, then, if we denote by $\mathcal{A} = \mathcal{R}/\mathcal{I}$ we have $\widehat{\mathcal{A}} = D_{\zeta_1} \oplus \cdots \oplus D_{\zeta_d}$ where $D_{\zeta_i} = Q_{\zeta_i}^{\perp}$, Q_{ζ_i} is the $\mathbf{m}_{\zeta_i}$-primary component of $\mathcal{I}$.*

We have $\mathcal{A}_{\zeta_i} = Q_{\zeta_i}/\mathcal{I}$ and we can show that $\widehat{\mathcal{A}}_{\zeta_i} = Q_{\zeta_i}^{\perp} \subset \mathcal{I}^{\perp}$. This basic fact allows us to compute the inverse system "locally", i.e. we look ζ_i as an isolated point of the variety and we can compute the inverse system associated to this point.

PROPOSITION 3.15: *Let Q_{ζ_i} be a $\mathbf{m}_{\zeta_i}$-primary component of $\mathcal{I}$, then $\mathcal{I}^{\perp} \subset \Theta_{\zeta_i}(\mathbb{K}[\partial])$.*

We recall that μ_i denote the multiplicity of ζ_i, we have the following theorem:

THEOREM 3.16: *Let $\mathcal{I} = (f_1, \ldots, f_n)$ such that $Z(\mathcal{I}) = \{\zeta_1, \ldots, \zeta_d\}$, for all $i \in \{1, \ldots, d\}$ there is a family $\left(\Lambda_1^{(i)}, \ldots, \Lambda_{\mu_i}^{(i)}\right)$ with $\Lambda_1^{(i)} = 1$, such that for all $\Lambda \in \mathcal{I}^{\perp}$ we have $\Lambda(\partial) = \sum_{i=1}^{d} e^{(\zeta_i, \partial)} \sum_{j=1}^{\mu_i} \Lambda_j^{(i)}(\partial)$. Furthermore if $f \in \mathcal{A}$, we have $\Lambda(f) = \sum_{i=1}^{d} \sum_{j=1}^{\mu_i} < \Lambda_j^{(i)}, f >_{\zeta_i}$.*

As a consequence we have that the family $\left(\Lambda_j^{(i)} e^{(\zeta_i, \partial)}\right)_{1 \le i \le d, 1 \le j \le \mu_i}$ is a basis of $\widehat{\mathcal{A}} = \mathcal{I}^{\perp}$. There are several algorithm to compute such a basis. So we will assume that we have the polynomials $\left(\Lambda_j^{(i)}\right)_{1 \le i \le d, 1 \le j \le \mu_i}$ as an input of our method. But we need to know how to compute a basis of $\mathcal{A}_{\zeta_i}$ thanks to the knowledge of those polynomials dual of that basis. A first simple remark is that $\left(Q_{\zeta_i}^{\perp}\right) = \widehat{\mathcal{A}}_{\zeta_i}$.

Now we know that $D_{\zeta_i} \subset \Theta(\mathbb{K}[[\partial_1, \ldots, \partial_n]])$, $\widehat{\mathcal{A}} = D_{\zeta_1} \oplus \cdots \oplus D_{\zeta_d}$ and $D_{\zeta_i} = \widehat{\mathcal{A}}_{\zeta_i}$ we can define the a bilinear form as follows :

DEFINITION 3.17: *The bilinear form* $< .,. >_{Z(\mathcal{I})}$: $\widehat{\mathcal{A}} \times \mathcal{A} \longrightarrow \mathbb{K}$ *is defined by associating to* $\Lambda = (\Lambda_1, \ldots, \Lambda_d) \in \widehat{\mathcal{A}}$ *and* $p \in \mathcal{A}$ *the scalar* $\sum_{i=1}^{d} < \Lambda_i, p >_{\zeta_i}$.

A fundamental property of that bilinear form is given by the following proposition :

PROPOSITION 3.18: *The bilinear form* $< .,. >_{\mathcal{Z}(\mathcal{I})}$ *is non degenerated.*

DEFINITION 3.19: *Let* $E \subset \mathbb{N}^n$*, we denote* $\mathbf{x}^E = \{\mathbf{x}^\alpha \mid \alpha \in E\}$ *and by* $\mathbb{K}[x_1, \ldots, x_n]_E$ *the set of polynomials with monomial support included in* $\mathbf{x}^E$*. We define the following application :*

$$\Pi_E : \left\{ \begin{array}{ccc} \mathbb{K}[\mathbf{x}, \mathbf{x}^{-1}] & \longrightarrow & \mathbb{K}[x_1, \ldots, x_n]_E \\ P = \sum_{\alpha \in \mathbb{Z}} p_\alpha \mathbf{x}^\alpha & \longmapsto & \Pi_E(P) = \sum_{\alpha \in E} p_\alpha \mathbf{x}^\alpha \end{array} \right.$$

PROPOSITION 3.20: *Let* $\Lambda(\partial) \in \widehat{\mathcal{A}}_{\zeta_i}$*, then we have the following formula :*

$$\forall P \in \mathcal{A}, \ \langle \Lambda(\partial), P \rangle_{\zeta_i} = \left\langle 1, \Pi_E \left(\Lambda\left((\mathbf{x} - \zeta_1)^{-1} \right) P \right) \right\rangle_{\zeta_i} \tag{6}$$

Let $\left(\Lambda_1^{(i)}, \ldots, \Lambda_{\mu_i}^{(i)}\right)$ be a basis of $\widehat{\mathcal{A}}_{\zeta_i}$ such that $\Lambda_1^{(i)} = 1$, the proposition 3.20 provides a computational tool to compute the dual basis of $\left(\Lambda_j^{(i)}\right)_{j \in \{1, \ldots, \mu_i\}}$ in $\mathcal{A}$ with the knowledge of an idempotent.

PROPOSITION 3.21: *Let* P *and* $Q \in \mathcal{A}$ *and* $\Lambda \in \widehat{\mathcal{A}}$*, then we have the following equality:*

$$\langle \Lambda(\partial), Q(\mathbf{x}) P(\mathbf{x}) \rangle_{\zeta_i} = \left\langle \Pi_{\mathbb{N}^n} \left(Q\left((\partial + \zeta_i)^{-1} \right) \Lambda(\partial) \right), P(\mathbf{x}) \right\rangle_{\zeta_i} \tag{7}$$

PROPOSITION 3.22: *Let* $\left(\Lambda_j^{(i)}\right)_{j \in \{1, \ldots, \mu_i\}}$ *(*$\Lambda_j^{(i)} \in \mathbb{K}[\partial]$*) be a basis of* $\widehat{\mathcal{A}}_{\zeta_i}$ *such that* $\Lambda_1^{(i)} = 1$*, then there exist polynomials* $\Psi_k^{(i)}$*,* $k \in \{1, \ldots, \mu_i\}$*, in* $\mathcal{A}$ *such that* $\Psi_1^{(i)} = 1$ *and* $\left\langle \Lambda_j^{(i)}, \Psi_k^{(i)} \mathbf{e}_{\zeta_i} \right\rangle_{\zeta_i} = \delta_{j,k}$.

Proof: First, one can remark that by definition $\left\langle \Lambda_j^{(i)}, \mathbf{e}_{\zeta_i} \right\rangle_{\zeta_i} = \delta_{1,j}$. Let us denote $(P_k)_{k \in \{1, \ldots, \mu_i\}}$ a dual basis of $\left(\Lambda_j^{(i)}\right)_{j \in \{1, \ldots, \mu_i\}}$. Then by proposition 3.7, one can show that there exist polynomials $\Psi_k^{(i)} \in \mathcal{A}$ such that $P_k = \Psi_k^{(i)} \mathbf{e}_{\zeta_i}$, $k \in \{1, \ldots, \mu_i\}$. $\square$

4. Multivariate interpolation

4.1. Introduction

The classical interpolation problem consists in finding polynomials with given values at fixed points $\zeta_1, \ldots, \zeta_d$ and with a fixed support. The general interpolation problem also called Hermite interpolation problem consists in computing the set of polynomials p satisfying tangential conditions $< \Lambda_j^{(i)}(\partial), p >_{\zeta_i} = v_{i,j}$ $j = 1, \ldots, \mu_i$ at fixed points $\zeta_1, \ldots, \zeta_d$, where $\Lambda_j^{(i)}(\partial)$ is a polynomial in the derivatives ∂. Hereafter, we will also denote by $\Lambda_j^{(i)}$ the corresponding linear form $p \mapsto < \Lambda_j^{(i)}(\partial), p >_{\zeta_i}$. The sequence of all linear forms $\Lambda_j^{(i)}$ will also be denoted by $\Lambda_1, \ldots, \Lambda_D$ where $D = \mu_1 + \cdots + \mu_d$. The problem can thus be reformulated as the problem of computing the orthogonal $(\Lambda_j^{(i)})^\perp$ and a complementary space in the space of polynomials. The basis of interpolation is the basis of this complementary space, which is dual to $(\Lambda_i)_{i=1,\ldots,D}$.

In order to be in an algebraic setting, we will consider hereafter the case where $(\Lambda_i)_{i=1,\ldots,D}^\perp$ *is an ideal of* $\mathcal{R}$. According to 3.12, it is equivalent to say that each set of polynomials $< \Lambda_j^{(i)}(\partial) >_{j=1,\ldots,\mu_i}$ is stable by derivation by ∂_{∂_i}. Hereafter the ideal $(\Lambda_i)^\perp$ is also denoted by I_Λ.

Under this condition we have the following property:

PROPOSITION 4.1: *The generalized interpolation problem* $\Lambda_i(p) = v_{i,j}$, $i = 1, \ldots, D$, *has a unique solution in* $< \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_D} >$ *if and only if the determinant of the matrix*

$$V_{\Lambda,E} = (\Lambda_i(\mathbf{x}^{\alpha_j}))_{i,j=1,\ldots,D}$$

is not zero.

This matrix allows us to solve the interpolation problem, using the following proposition:

PROPOSITION 4.2: *Assume that the matrix* $V_{\Lambda,E}$ *is invertible and let* $W_{\Lambda,E} = V_{\Lambda,E}^{-1}$. *Then the representents, in the basis* $\{\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_D}\}$, *of the dual basis of* $(\Lambda_i)_{i=1,\ldots,D}$ *are*

$$\mathbf{w}_i = \sum_i W_{i,j}\, \mathbf{x}^{\alpha_j}.$$

Proof: Let $\mathbf{w}_i$ be these representents in the basis $\{\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_D}\}$ of $\mathcal{A}_\Lambda$. They satisfy $\Lambda_i(\mathbf{w}_j) = \delta_{i,j}$. Denoting by $w_{i,j}$ their coefficients with respect to the basis $\mathbf{x}^{\alpha_j}$ ($\mathbf{w}_j = \sum_{k=1}^D w_{j,k}\mathbf{x}^{\alpha_k}$) we have $\Lambda_i(\mathbf{w}_j) = \sum_{k=1}^D w_{j,k}\Lambda_i(\mathbf{x}^{\alpha_k})$, which means that the matrix $(w_{j,k})$ is the inverse of $V_{\Lambda,E}$. □

This allows us to answer directly to the interpolation problem.

THEOREM 4.3: *The interpolation problem* $\Lambda_j(p) = v_j$ *for* $j = 1, \ldots, D$ *has a* *unique solution in the vector space* $< \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_D} >$ *if and only if* $det(V_{\Lambda,E}) \neq 0$. *In such a case, the solution is given by*

$$p = \sum_{i=1}^{D} v_i \, \mathbf{w}_i.$$

where $\mathbf{w}_i = \sum_i W_{i,j} \, \mathbf{x}^{\alpha_j}$ *and* $(W_{i,j})$ *is the inverse matrix of* $V_{\Lambda,E}$.

Proof: We check directly that $\Lambda_i(p) = v_i$ and that $\mathbf{w}_i \in < \mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_D} >$. $\square$

We are going now to give an explicit formula for such a solution in terms of the monomials $\mathbf{x}^{\alpha_i}$ and the linear forms Λ_j.

4.2. Simple roots

Let us detail first, the case of *simple roots* $\mathbf{z} = \{\zeta_1, \ldots, \zeta_D\}$, for which the linear forms Λ_i are the evaluations 1_{ζ_i}, $i = 1 \ldots, d$ are the roots $\zeta_i \in \mathbb{K}^n$. We easily check that $< 1_{\zeta_i} >$ is stable by derivation since $x_i \cdot 1_{\zeta_j} = \zeta_{j,i} \, 1_{\zeta_j}$ so that $(\Lambda)^{\perp}$ is an ideal (indeed the ideal of polynomials vanishing at the points $\zeta_1, \ldots, \zeta_d$). Let us denote it by $I_{\mathbf{z}}$ and by $\mathcal{A}_{\mathbf{z}} = \mathcal{R}/I_{\zeta}$ the quotient algebra.

By proposition 3.4, the local ring associated with ζ_i is $\mathcal{A}_{\zeta_i} = \mathbb{K}\mathbf{e}_{\zeta_i}$ and we obtain the following proposition :

PROPOSITION 1: *The quotient algebra* $\mathcal{A}_{\mathbf{z}}$ *is a* d*-dimensional* $\mathbb{K}$*-vector space that admit the set of idempotents* $(\mathbf{e}_{\zeta_i})_{i \in \{1,\ldots,d\}}$ *as basis. Its dual basis in* $\widehat{\mathcal{A}}$ *is* $(1_{\zeta_i})_{i \in \{1,\ldots,d\}}$.

This proposition is in fact a consequence of the Cauchy formula: $\forall a \in \mathcal{A}$,

$$a = \sum_{i=1}^{d} 1_{\zeta_i}(a) \, \mathbf{e}_{\zeta_i} = \sum_{i=1}^{d} a(\zeta_i) \, \mathbf{e}_{\zeta_i}.$$

which also yields an interpolation formula, generalizing the Lagrange interpolation formula. This representation of $\mathcal{A}_{\mathbf{z}}$ has several advantages. One of them is the simplicity of the multiplicative structure. Indeed, $\forall a$ and $b \in \mathcal{A}$ we have

$$ab = \sum_{i=1}^{d} 1_{\zeta_i}(ab) \, \mathbf{e}_{\zeta_i} = \sum_{i=1}^{d} a(\zeta_i) \, b(\zeta_i) \, \mathbf{e}_{\zeta_i}.$$

DEFINITION 4.4: *Let* $\mathbf{z} = \{\zeta_1, \ldots, \zeta_d\}$. *Let* $E = \{\alpha_1, \ldots, \alpha_d\} \subset \mathbb{N}^n$, *we denote* $\mathbf{x}^E = \{\mathbf{x}^{\alpha} | \alpha \in E\}$. *We call the Vandermonde's matrix associated to* E *and* $\{\zeta_1, \ldots, \zeta_d\}$ *the following matrix :*

$$V_{\mathbf{z},E} = \begin{pmatrix} \zeta_1^{\alpha_1} & \cdots & \zeta_1^{\alpha_d} \\ \vdots & \ddots & \vdots \\ \zeta_d^{\alpha_1} & \cdots & \zeta_d^{\alpha_d} \end{pmatrix} \tag{8}$$

The Vandermonde's determinant associated to E *and* $\mathbf{z}$ *is* $\mathbf{v}_{\mathbf{z},E} = det(V_{\mathbf{z},E})$.

According to proposition 4.1, we have:

PROPOSITION 2: *Let* $\mathbf{z} = \{\zeta_1, \ldots, \zeta_d\}$ *and let* $E = \{\alpha_1, \ldots, \alpha_d\} \subset \mathbb{N}^n$. *The set* $\mathbf{x}^E$ *is monomial basis of* $\mathcal{A}_\mathbf{z} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_\mathbf{z}$ *if and only if the Vandermonde's determinant associated to* E *and* $\mathbf{z}$ *is not zero.*

We denote by $\mathbb{K}[x_1, \ldots, x_n]_E = \left\{ p \in \mathbb{K}[x_1, \ldots, x_n] \,|\, p = \sum_{\alpha \in E} p_\alpha \mathbf{x}^\alpha \right\}$, i.e. the set of polynomials with monomial support in $\mathbf{x}^E$.

DEFINITION 4.5: *Let* $\mathbf{z} = \{\zeta_1, \ldots, \zeta_d\}$ *and let* $E = \{\alpha_1, \ldots, \alpha_d\} \subset \mathbb{N}^n$ *such that* $\det(V_{\mathbf{z},E}) \neq 0$ *(*$\mathbf{x}^E$ *is a monomial basis of* $\mathcal{A}_\mathbf{z}$*). We define the pseudo Lagrange's polynomial associated with* ζ_i *by:*

$$\mathbf{e}_i = -\frac{1}{V_{\mathbf{z},E}} \begin{vmatrix} \mathbf{x}^{\alpha_1} & \cdots & \mathbf{x}^{\alpha_d} \\ \zeta_1^{\alpha_1} & \cdots & \zeta_1^{\alpha_d} \\ \vdots & & \vdots \\ \zeta_{i-1}^{\alpha_1} & \cdots & \zeta_{i-1}^{\alpha_d} \\ \zeta_{i+1}^{\alpha_1} & \cdots & \zeta_{i+1}^{\alpha_d} \\ \vdots & & \vdots \\ \zeta_d^{\alpha_1} & \cdots & \zeta_d^{\alpha_d} \end{vmatrix} \tag{9}$$

PROPERTIES 4.6: *The class of* $\mathbf{e}_i$ *in* $\mathcal{A}$ *is the idempotent associated with the root* ζ_i *and* $\mathbf{e}_i \in \mathbb{K}[x_1, \ldots, x_n]_E$, *for* $i = 1, \ldots, d$.

Proof: Let us denote by $\mathbf{e}'_i$ the determinant polynomial defined in (9). We check that $\mathbf{e}'_i(\zeta_j) = \delta_{i,j}$, so that according to proposition 3.5, $\mathbf{e}_i \equiv \mathbf{e}'_i$. □

This property yields the solution to the interpolation problem as follows:

THEOREM 4.7: *Let us assume that* $\det(V_{\mathbf{z},E})$ *is not zero. Then for any value* $v_i \in \mathbb{K}$, *there exists a unique polynomial* $p \in \mathbb{K}[x_1, \ldots, x_n]_E$ *such that* $p(\zeta_i) = v_i$. *It is given by*

$$p(\mathbf{x}) = \sum_{i=1}^{d} v_i \, \mathbf{e}_i.$$

4.3. Roots with multiplicities

We now describe explicitly the general case of Hermite interpolation. Let us consider μ_i tangential conditions $\Lambda_{i,j}(p) = v_i$ at points ζ_i, for $i = 1, \ldots, d$. They are of the form

$$< \Lambda_j^{(i)}(\delta), p >_{\zeta_i} = v_i$$

where $\Lambda_j^{(i)}(\delta)$ is a polynomial in δ.

We assume that each $(\Lambda_j^{(i)})_{j=1,\ldots,\mu_i}$ is stable by derivation so that $(\Lambda)^\perp =$

$(\Lambda_{i,j})^{\perp} := I_{\Lambda}$ is an ideal of R. We denote by $\mathcal{A}_{\Lambda} = \mathcal{R}/I_{\Lambda}$ the associated quotient algebra.

The dual space associated to the local algebra $\mathcal{A}_{\zeta_i}$ is $(\Lambda_j^{(i)})_{j=1,\ldots,\mu_i}$. Because these elements are polynomials in ∂_i,and the vector space is stable by derivation by ∂_{∂_i}, it contains the element 1 We assume, without lost of generality, that $lfi1 = 1$ *and that the* $\Lambda_j^{(i)}(\partial)$ *have no constant terms* (it can be done by substraction of a scalar multiple of $\Lambda_1^{(i)} = 1$). We define a generalization of the Vandermonde matrix.

DEFINITION 4.8: *With the above assumptions and notations, we define the Vandermonde's matrix associated to E and $Z = (\zeta_1, \ldots, \zeta_d)$ the following matrix:*

$$V_{\Lambda,E} = \begin{pmatrix} < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & & \vdots \\ < \Lambda_{\mu_1}^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_{\mu_1}^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & & \vdots \\ \vdots & & \vdots \\ < \Lambda_1^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_1^{(d)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \\ \vdots & & \vdots \\ < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_{\mu_{\zeta_d}}^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \end{pmatrix}$$

The Vandermonde's determinant associated to $Z(\mathcal{I})$ and E is $\mathbf{v}_{\Lambda,E} = det(V_{\Lambda,E})$

According to proposition 4.1, we have:

THEOREM 4.9: *Let $Z = \{\zeta_1, \ldots, \zeta_d\} \in \mathbb{K}^n$ and let $E = \{\alpha_1, \ldots, \alpha_D\} \subset \mathbb{N}^n$. Therefore, $\mathbf{x}^E$ is a basis of $\mathcal{A}_{\Lambda} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{\Lambda}$ if and only if the Vandermonde's determinant* $\det(V_{\Lambda,E}) \neq 0$.

DEFINITION 4.10: *Let $Z = \{\zeta_1, \ldots, \zeta_d\} \subset \mathbb{K}^n$ and let $E = \{\alpha_1, \ldots, \alpha_D\} \subset \mathbb{N}^n$ such that $\mathbf{x}^E$ is a basis of $\mathcal{A}_{\Lambda} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}_{\Lambda}$. We define the pseudo-Lagrange's polynomials $\mathbf{e}_i^{(j)}, j = 1, \ldots, \mu_i$ associated to the root ζ_i in the following* *way:*

$$\mathbf{e}_i^{(j)} = -\frac{1}{V_{\Lambda,E}} det \begin{pmatrix} \mathbf{x}^{\alpha_1} & \cdots & \mathbf{x}^{\alpha_D} \\ <\Lambda_1^{(1)}, \mathbf{x}^{\alpha_1}>_{\zeta_1} & \cdots & <\Lambda_1^{(1)}, \mathbf{x}^{\alpha_D}>_{\zeta_1} \\ \vdots & & \vdots \\ <\Lambda_{\mu_1}^{(1)}, \mathbf{x}^{\alpha_1}>_{\zeta_1} & \cdots & <\Lambda_{\mu_{\zeta_1}}^{(1)}, \mathbf{x}^{\alpha_D}>_{\zeta_1} \\ \vdots & & \vdots \\ <\Lambda_1^{(i)}, \mathbf{x}^{\alpha_1}>_{\zeta_i} & \cdots & <\Lambda_1^{(i)}, \mathbf{x}^{\alpha_D}>_{\zeta_i} \\ \vdots & & \vdots \\ <\Lambda_{j-1}^{(i)}, \mathbf{x}^{\alpha_i}>_{\zeta_i} & \cdots & <\Lambda_{j-1}^{(i)}, \mathbf{x}^{\alpha_D}>_{\zeta_i} \\ <\Lambda_{j+1}^{(i)}, \mathbf{x}^{\alpha_1}>_{\zeta_1} & \cdots & <\Lambda_{j+1}^{(i)}, \mathbf{x}^{\alpha_D}>_{\zeta_1} \\ \vdots & & \vdots \\ <\Lambda_{\mu_i}^{(i)}, \mathbf{x}^{\alpha_1}>_{\zeta_i} & \cdots & <\Lambda_{\mu_i}^{(i)}, \mathbf{x}^{\alpha_D}>_{\zeta_i} \\ \vdots & & \vdots \\ <\Lambda_1^{(d)}, \mathbf{x}^{\alpha_1}>_{\zeta_d} & \cdots & <\Lambda_1^{(1)}, \mathbf{x}^{\alpha_D}>_{\zeta_d} \\ \vdots & & \vdots \\ <\Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1}>_{\zeta_d} & \cdots & <\Lambda_{\mu_d}^{(1)}, \mathbf{x}^{\alpha_D}>_{\zeta_d} \end{pmatrix}$$

For all $i \in \{1, \ldots, d\}$ *and* $j \in \{1, \ldots, \mu_i\}$.

PROPOSITION 4.11: *The polynomial* $\mathbf{e}_{\zeta_i}^{(1)}$ *is the idempotent associated with* ζ_i *and* $\mathbf{e}_i^{(1)}, \ldots, \mathbf{e}_i^{(\mu_i)}$ *is a basis of the local ring* $\mathcal{A}_{\zeta_i}$.

Proof: We remark that

$$<\Lambda_j^{(i)}, \mathbf{e}_{i'i}^{(j')}>_{\zeta_i} = \det \begin{pmatrix} <\Lambda_j^{(i)}, \mathbf{x}^{\alpha_1}>_{\zeta_i} & \cdots & <\Lambda_j^{(i)}, \mathbf{x}^{\alpha_D}>_{\zeta_i} \\ <\Lambda_2^{(i)}, \mathbf{x}^{\alpha_1}>_{\zeta_i} & \cdots & <\Lambda_2^{(i)}, \mathbf{x}^{\alpha_D}>_{\zeta_i} \\ \vdots & & \vdots \\ <\Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1}>_{\zeta_i} & \cdots & <\Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_D}>_{\zeta_i} \end{pmatrix} = \delta_{i,i'}\delta_{j,j'}$$

Therefore the elements $\mathbf{e}_i^{(j)}$ $j = 1, \ldots, \mu_i$ are the dual basis of the basis $\Lambda_j^{(i)}$ of $\widehat{\mathcal{A}}$. More over, as $\mathbf{e}_i^{(1)}$ is the dual element of $\Lambda_1^{(i)} = 1$ and because $\Lambda_j^{(i)}$ $j = 2, \ldots, \mu_i$ have no constant term $\mathbf{e}_i^{(j)}$ $j = 2, \ldots, \mu_i$ are in the maximal ideal $\mathbf{m}_{\zeta_i}$ of $\mathcal{A}_{\zeta_i}$. Therefore $\mathbf{e}_i^{(1)}$ is the idempotents associated with ζ_i. □

This yields the solution to the generalized Hermite interpolation problem:

PROPOSITION 4.12: *With the above assumptions and notations, for all* $f \in \mathcal{A}$, *we have:*

$$f = \sum_{i=1}^{d} \sum_{j=1}^{\mu_i} v_{i,j}\, \mathbf{e}_i^{(j)}$$

where $\mathbf{e}_i^{(j)}$ *is the polynomial of definition 4.10,* $v_{i,j} = <\Lambda_j^{(i)}, f>_{\zeta_i}$, *for* $i = 1, \ldots, d$ *and* $j = 1, \ldots, \mu_i$.

In order to answer completely to the interpolation problem, we need to describe the polynomials for which the tangential condition vanish, that is the polynomials of I_Λ. Just as for the idempotents, their will described by determinantal expressions. Let Q be a polynomial of $\mathbb{K}[x_1,\ldots,x_n]$. We denote by $N_Q(\mathbf{x}) = \sum_{\alpha\in E} n_{Q,\alpha}\mathbf{x}^\alpha$ the normal form of the polynomial Q in the basis $\mathbf{x}^E$. We consider the polynomial $P_Q(\mathbf{x}) = Q(\mathbf{x}) - N_Q(\mathbf{x})$. Then $P_Q \in \mathcal{I}$.

Definition 4.13: *We define the polynomial $R_Q(Z(\mathcal{I}),\mathbf{x})$ as the following determinant:*

$$R_Q(\Lambda,\mathbf{x}) = \begin{vmatrix} Q(\mathbf{x}) & \mathbf{x}^{\alpha_1} & \cdots & \mathbf{x}^{\alpha_D} \\ < \Lambda_1^{(1)}, Q(\mathbf{x}) >_{\zeta_1} & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_{\mu_d}^{(d)}, Q(\mathbf{x}) >_{\zeta_d} & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \end{vmatrix} \tag{10}$$

We check that we have the following properties:

Properties 4.14:

- $R_Q(\Lambda,\zeta_i) = 0$,
- $R_Q(\Lambda,\mathbf{x}) = V_{\Lambda,E} P_Q(\mathbf{x})$.

We now define a function associated to this relation:

Definition 4.15: *We define the following polynomial:*

$$F_Q(\Lambda,\mathbf{x}) = \frac{R_Q(\Lambda,\mathbf{x})}{V_{\Lambda,E}} - P_Q(\mathbf{x}) \tag{11}$$

For all $i \in \{1,\ldots,n\}$, we denote by $\nu_i = (\delta_{i,j})_{j\in\{0,\ldots,n\}} \in \mathbb{N}^n$. Let $Q \notin \mathbb{K}[x_1,\ldots,x_n]_E$. The partial derivative functions of R_Q relatively to the coordinates of ζ_i have a strong property that we will describe.

Proposition 4.16: *We have:*

$$\frac{\partial}{\partial\zeta_{i,j}} R_Q(\Lambda,\mathbf{x}) \in \mathcal{A}_{\zeta_i}. \tag{12}$$

Proof: We consider the determinantal expression of this polynomial:

$$\frac{\partial}{\partial\zeta_{i,j}} R_Q(Z(\mathcal{I},\mathbf{x})) = \frac{\partial}{\partial\zeta_{i,j}} \begin{vmatrix} Q(\mathbf{x}) & \mathbf{x}^{\alpha_1} & \cdots & \mathbf{x}^{\alpha_D} \\ < \Lambda_1^{(1)}, Q(\mathbf{x}) >_{\zeta_1} & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_{\mu_d}^{(d)}, Q(\mathbf{x}) >_{\zeta_d} & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \end{vmatrix}$$

So we have the following equality :

$$\frac{\partial}{\partial \zeta_{i,j}} R_Q(\Lambda, \mathbf{x}) = \begin{vmatrix} Q(\mathbf{x}) & \mathbf{x}^{\alpha_1} & \cdots & \mathbf{x}^{\alpha_D} \\ < \Lambda_1^{(1)}, Q(\mathbf{x}) >_{\zeta_1} & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_1^{(i)}, \frac{\partial}{\partial \zeta_{i,j}} Q(\mathbf{x}) >_{\zeta_i} & < \Lambda_1^{(i)}, \frac{\partial}{\partial \zeta_{i,j}} \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_1^{(i)}, \frac{\partial}{\partial \zeta_{i,j}} \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_{\mu_i}^{(i)}, \frac{\partial}{\partial \zeta_{i,j}} Q(\mathbf{x}) >_{\zeta_i} & < \Lambda_{\mu_i}^{(i)}, \frac{\partial}{\partial \zeta_{i,j}} \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_{\mu_i}^{(i)}, \frac{\partial}{\partial \zeta_{i,j}} \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_{\mu_d}^{(d)}, Q(\mathbf{x}) >_{\zeta_d} & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \end{vmatrix}$$

Then we remark that $\frac{\partial}{\partial \zeta_{i,j}} R_Q(Z(\mathcal{I}, \zeta_k)) = 0$ if $k \neq i$. Then $\frac{\partial}{\partial \zeta_{i,j}} R_Q(Z(\mathcal{I}, \zeta_k)) \in \mathcal{A}_{\zeta_i}$. □

Corollary 4.17:

$$\frac{\partial}{\partial \zeta_{i,j}} R_Q(\Lambda, \mathbf{x}) = \sum_{j=1}^{\mu_{\zeta_i}} < \Lambda_j^{(i)}, \frac{\partial}{\partial x_j} R_Q(\Lambda, \mathbf{x}) >_{\zeta_i} \mathbf{e}_{\zeta_i}^j \tag{13}$$

Corollary 4.18: *If ζ_i is a simple root, then:*

$$\frac{\partial}{\partial \zeta_{i,j}} R_Q(\Lambda, \mathbf{x}) =< 1, \frac{\partial}{\partial x_j} R_Q(\Lambda, \mathbf{x}) >_{\zeta_i} \mathbf{e}_{\zeta_i} = \frac{\partial}{\partial x_j} R_Q(\Lambda, \zeta_i) \mathbf{e}_{\zeta_i}. \tag{14}$$

4.4. Relations between roots and coefficients

In this section we give some relations between the coefficients of an algebraic system that defines a zero-dimensional complete intersection and the roots of the system. Those relations extend naturally the well known relations between elementary symmetric functions of the roots and the coefficients of an univariate equation.

Let $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$ such that $\mathcal{I} = (f_1, \ldots, f_n)$ is a zero-dimensional complete intersection. We denote by $Z = \{\zeta_1, \ldots, \zeta_d\}$ the associated algebraic set and $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$ its coordinate algebra. We assume that we know a monomial basis $\mathbf{x}^E$ of $\mathcal{A}$ and a basis $\left(\Lambda_j^{(i)}\right)_{j \in \{1,\ldots \mu_i\}}$ of the dual space of each local algebra $\mathcal{A}_{\zeta_i i}$. Those basis are such that $\Lambda_1^{(i)} = 1$, $\forall i \in \{1, \ldots, d\}$.

Let Q be a polynomial of $\mathbb{K}[x_1, \ldots, x_n]$, we denote by $N_Q(\mathbf{x}) = \sum_{\alpha \in E} n_{Q,\alpha} \mathbf{x}^\alpha$ the normal form of Q in $\mathcal{A}$ with respect to the basis $\mathbf{x}^E$. We recall that a property of 4.14 gives $\frac{R_{f_i}(Z, \mathbf{x})}{\mathbf{v}_{Z,E}} = f_i(\mathbf{x})$ for all $i \in \{1, \ldots, d\}$. We decompose each $f_i = \sum_{\alpha \in \mathbb{N}^n} f_{i,\alpha} \mathbf{x}^\alpha$ in the following way : $f_i(\mathbf{x}) = \overline{f_i} + \sum_{\alpha \in E} f_{i,\alpha} \mathbf{x}^\alpha$ with $\overline{f_i}$ with monomial support disjoint of $\mathbf{x}^E$. We have $\frac{\mathbb{R}_{\overline{f_i}}(Z,E)}{\mathbf{v}_{Z,E}} = \overline{f_i}\,(\mathbf{x}) - N_{\overline{f_i}}\,(\mathbf{x}) = f_i\,(\mathbf{x})$ since $N_{f_i}\,(\mathbf{x}) = 0$ and so $\sum_{\alpha \in E} f_{i,\alpha}\mathbf{x}^\alpha = N_{\overline{f_i}}$. So we get $\frac{R_{\overline{f_i}}(Z,\mathbf{x})}{\mathbf{v}_{Z,E}} - \overline{f_i} = \sum_{\alpha \in E} f_{i,\alpha}\mathbf{x}^\alpha$. We recall that by definition we have the following equality :

$$R_{\overline{f_i}}\,(\Lambda,\mathbf{x}) = \begin{vmatrix} \overline{f_i}\,(\mathbf{x}) & \mathbf{x}^{\alpha_1} & \cdots & \mathbf{x}^{\alpha_D} \\ < \Lambda_1^{(1)}, \overline{f_i}\,(\mathbf{x}) >_{\zeta_1} & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_1^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_{\mu_1}^{(1)}, \overline{f_i}\,(\mathbf{x}) >_{\zeta_1} & < \Lambda_{\mu_1}^{(1)}, \mathbf{x}^{\alpha_1} >_{\zeta_1} & \cdots & < \Lambda_{\mu_1}^{(1)}, \mathbf{x}^{\alpha_D} >_{\zeta_1} \\ \vdots & \vdots & & \vdots \\ < \Lambda_1^{(d)}, \overline{f_i}\,(\mathbf{x}) >_{\zeta_d} & < \Lambda_1^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_1^{(d)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \\ \vdots & \vdots & & \vdots \\ < \Lambda_{\mu_d}^{(d)}, \overline{f_i}\,(\mathbf{x}) >_{\zeta_d} & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_1} >_{\zeta_d} & \cdots & < \Lambda_{\mu_d}^{(d)}, \mathbf{x}^{\alpha_D} >_{\zeta_d} \end{vmatrix} \tag{15}$$

We denote by $\left|R_{\overline{f_i}}\,(\Lambda,\mathbf{x})\right|_{\alpha_i}$ the minor of the matrix obtain by removing the first row and the column corresponding to $\mathbf{x}^{\alpha_i}$, $\forall i \in \{1,\dots,D\}$. The following proposition is, now, easy to check :

Theorem 4.19: *Let $i \in \{1,\dots,D\}$, we have* $\frac{\left|R_{\overline{f_i}}(\Lambda,\mathbf{x})\right|_{\alpha_i}}{\mathbf{v}_{\Lambda,E}} = f_{i,\alpha}$.

The equality of this proposition generalizes the well know relation between elementary symmetric functions and coefficients of an univariate algebraic equation.

4.5. Relation polynomials and interpolation

We consider the following Hermite interpolation problem: We consider a n algebraic set given by the points of the set $Z = \{\zeta_1,\dots,\zeta_d\}$, the structure of the dual space thanks to differential polynomials $\left(\Lambda_j^{(i)}\right)_{j\in\{1,\dots,\mu_i\}}$ those generate the dual space of the local algebra associated to ζ_i, $i \in \{1,\dots,d\}$ and a monomial basis $\mathbf{x}^E$ of the coordinate algebra. Find an algebraic system that defines a quotient algebra $\mathcal{A}$ with $\mathbf{x}^E$ as a monomial basis, the set $\{\zeta_1,\dots,\zeta_d\}$ as set of solutions and such that $\mathcal{A}_{\zeta_i} =< \Lambda_1^{(i)},\dots,\Lambda_{\mu_i}^{(i)} >^\perp$ for all $i \in \{1,\dots,d\}$. In the case of simple roots, the problem is to find a set of polynomials that vanish at $\zeta_i,\dots,\zeta_d$. We have the same conditions on that $\Lambda_j^{(i)}$ and $\mathbf{x}^E$ than the previous Hermite interpolation problem (stability by derivation, non vanishing of the Vandermonde determinant).

Definition 4.20:
We denote $\Omega = \left\{\beta \in \mathbb{N}^n \,|\, \exists\alpha \in E \text{ and } k \in \{1,\dots,n\},\, x^\beta = x_k * \mathbf{x}^\alpha \text{ and } \beta \notin E\right\}$.
We call this set the frontier of the quotient algebra.

PROPOSITION 4.21: *The set formed by the polynomials* $R_{\mathbf{x}^\beta}(\Lambda, \mathbf{x})$, $\beta \in \Omega$, *is a solution of the Hermite interpolation problem, i.e. if* $\mathcal{I} = (R_{\mathbf{x}^\beta}(\Lambda, \mathbf{x}))_{\beta \in \Omega}$ *is such that* $Z(\mathcal{I}) = \{\zeta_1, \ldots, \zeta_d\}$, $\mathbf{x}^E$ *is a basis of* $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$ *as* $\mathbb{K}$*-vector space and for all* $i \in \{1, \ldots, d\}$, $\mathcal{A}_{\zeta_i} =< \Lambda_1^{(i)}, \ldots, \Lambda_{\mu_i}^{(i)} >^\perp$.

Proof: First we show that $\mathbf{x}^E$ is a basis of $\mathcal{A}$.If $\gamma \in \mathbb{N}^n$ is such that $\mathbf{x}^\gamma$ not in E then we can reduce it by an $\mathbf{x}^\beta$, $\beta \in \Omega$. So $\mathbf{x}^E$ is a generating set of $\mathcal{A}$. Because the Vandermonde determinant $\mathbf{v}_{\lambda,E} \neq 0$, it a free family. Finally $\mathbf{x}^E$ is a basis of $\mathcal{A}$.

We have $Z \subset Z(\mathcal{I})$ by definitions of the relation polynomials. Furthermore, always by definition, $\left\langle \Lambda_j^{(i)}, R_{\mathbf{x}^\beta}(\Lambda, \mathbf{x}) \right\rangle_{\zeta_i} = 0$. So ζ_i as at least the multiplicity μ_i. Because $\mathbf{x}^E$ is a basis of $\mathcal{A}$ and that $\sum_{i=1}^{d} \mu_i = D = \#(E)$ then the multiplicity is exactly μ_i. □

With this approach, we generally have more equation than variables. If we want as many equations as variables, some polynomials those define a bigger algebraic set, i.e. the wanted coordinate algebra is a sub-algebra of the computed one. But if we search a basis for a projective zero-dimensional complete intersection without roots at infinity, then we can certify that the two algebras match.

This interpolation process is used at several levels in the global algebraic solver exposed in the next section.

5. Weierstrass's method

In this section we give a generalization of the Weierstrass's method for the multivariate for a complete intersection that defines only simple roots. We construct an iteration function by a direct extension of the methodology exposed in section 2. We do that thanks to the representation of the quotient algebra given in section 3 and 4. The iteration function gives a local method with a quadratic convergence. To obtain a global method, we will use this local method in a continuation process.

Hereafter $\mathcal{I} = (f_1, \ldots, f_n)$ is a complete intersection zero-dimensional ideal of $\mathbb{K}[x_1, \ldots, x_n]$ that defines only simple roots $Z(\mathcal{I}) = \{\zeta_1, \ldots, \zeta_D\}$.

HYPOTHESIS 5.1:
We assume that we know a basis a set $E = \{\alpha_1, \ldots, \alpha_D\} \subset \mathbb{N}^n$ *such that* $\mathbf{x}^E$ *is a monomial basis of* $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$.*(This can be done by the computation of a Grobner basis using modular arithmetic with a high probability of success for instance.)*

5.1. Iteration function

We begin by the construction of an equivalent system to the initial algebraic system. We define the following application :

$$\mathcal{F}:\left\{\begin{array}{ccc}\left(\mathbb{K}^{n}\right)^{D} & \longrightarrow & \left(\mathbb{K}[x_1,\ldots,x_n]_E\right)^{n}\\ \mathbf{z} & \longmapsto & \mathcal{F}\left(\mathbf{z},\mathbf{x}\right)=\begin{pmatrix}F_{f_1}\\ \vdots\\ F_{f_n}\end{pmatrix}\end{array}\right. \tag{16}$$

We can associate the equation $\mathcal{F}\left(\mathbf{z}\right)=0$ a system that will described now. If we look at the polynomial $F_{f_i}\left(\mathbf{z},\mathbf{x}\right)=\frac{R_{f_i}(\mathbf{z},\mathbf{x})}{\mathbf{v}_{Z(\mathcal{I}),E}}-f_i$, we can remark that he has $\mathbf{x}^E$ as monomial support, $\forall i\in\{1,\ldots,n\}$. So we have $F_{f_i}\left(\mathbf{z},\mathbf{x}\right)=\sum_{\alpha\in E}\psi_{i,\alpha}\left(\mathbf{z}\right)\mathbf{x}^{\alpha}$, $\forall i\in\{1,\ldots,n\}$. Then we obtain the following square $nD\times nD$ system :

$$(\Psi)\left\{\begin{array}{c}\psi_{i,\alpha}\left(\mathbf{z}\right)=0\\ \forall i\in\{1,\ldots,n\}\text{ and }\alpha\in E\end{array}\right. \tag{17}$$

By construction we have $\psi_{i,\alpha}\left(Z\left(\mathcal{I}\right)\right)=0$, $\forall i\in\{1,\ldots,n\}$ and $\alpha\in E$. The system 17 is the analogous of the system 1 for the multivariate case. The idea is again to apply the Newton's method to the system 17. As in the univariate case, we will exploit the structure of the problem to inverse explicitly the jacobian of the system and give a simple iteration function.

5.1.1. Computation of the iteration function

We consider the following maps:

- $\Phi:\left\{\begin{array}{ccc}\mathbb{K}[x_1,\ldots,x_n]_E & \longrightarrow & \mathbb{K}^D\\ P=\sum_{\alpha\in E}p_{\alpha}\mathbf{x}^{\alpha} & \longmapsto & \left(p_{\alpha}\right)_{\alpha\in E}\end{array}\right.$
- $\overrightarrow{\mathcal{F}}:\left\{\begin{array}{ccc}\left(\mathbb{K}^{n}\right)^{D} & \longrightarrow & \left(\mathbb{K}^{D}\right)^{n}\\ \mathbf{z} & \longmapsto & \overrightarrow{\mathcal{F}}\left(\mathbf{z}\right)=\left(\Phi\left(F_{f_1}\right),\ldots,\Phi\left(F_{f_n}\right)\right)\end{array}\right.$

By proposition ? we have that $\overrightarrow{\mathcal{F}}\left(\mathbf{z}\right)=0$ if and only if $\{\mathbf{z}_1,\ldots,\mathbf{z}_d\}=\{\zeta_1,\ldots,\zeta_d\}$.

The idea is to apply the Newton's method to $\overrightarrow{\mathcal{F}}$ and to do that we need to compute the sequence of Newton's iteration points:

$$\mathbf{z}^{(k+1)}=\mathbf{z}^{(k)}-J_{\overrightarrow{\mathcal{F}}}\left(\mathbf{z}^{(k)}\right)^{-1}\overrightarrow{\mathcal{F}}\left(\mathbf{z}^{(k)}\right) \tag{18}$$

THEOREM 5.2: *In the above situation, to apply a Newton's iteration at $\overrightarrow{\mathcal{F}}$, for*

$i \in \{1, \ldots, D\}$, *we only need to compute*

$$\mathbf{z}_i^{(k+1)} = \mathbf{z}_i^{(k)} - \begin{pmatrix} \frac{\frac{\partial R_{f_1}}{\partial x_1}\left(\mathbf{z}^{(k)},\mathbf{z}_i^{(k)}\right)}{\mathbf{v}_{\mathbf{z}^{(k)},E}} & \cdots & \frac{\frac{\partial R_{f_1}}{\partial x_n}\left(\mathbf{z}^{(k)},\mathbf{z}_i^{(k)}\right)}{\mathbf{v}_{\mathbf{z}^{(k)},E}} \\ \vdots & \ddots & \vdots \\ \frac{\frac{\partial R_{f_n}}{\partial x_1}\left(\mathbf{z}^{(k)},\mathbf{z}_i^{(k)}\right)}{\mathbf{v}_{\mathbf{z}^{(k)},E}} & \cdots & \frac{\frac{\partial R_{f_n}}{\partial x_n}\left(\mathbf{z}^{(k)},\mathbf{z}_i^{(k)}\right)}{\mathbf{v}_{\mathbf{z}^{(k)},E}} \end{pmatrix}^{-1} \begin{pmatrix} P_{f_1}\left(\mathbf{z}_i^{(k)}\right) \\ \vdots \\ P_{f_n}\left(\mathbf{z}_i^{(k)}\right) \end{pmatrix} \tag{19}$$

And the sequence $\left(\mathbf{z}^{(k)}\right)_{k\in\mathbb{N}}$ *converge quadratically in a neighborhood of the solutions of the system.*

Remark 5.3: *Because of the linearity of* Φ *we have*
$\frac{\partial}{\partial z_{i,j}}\Phi\left(F_{f_k}(\mathbf{z})\right) = \Phi\left(\frac{\partial}{\partial z_{i,j}}F_{f_k}(\mathbf{z})\right)$ *for* j *and* $k \in \{1, \ldots, n\}$ *and* $i \in \{1, \ldots, D\}$.

Proof: Let us solve $J_{\overrightarrow{\mathcal{F}}}(\mathbf{z})\,\mathbf{u} = \overrightarrow{\mathcal{F}}(\mathbf{z})$. Thanks to corollary (4.18), for all $k \in \{1, \ldots, n\}$ we have

$$\Phi\left(\sum_{i=1}^{D}\sum_{j=1}^{n} -\frac{\frac{\partial R_{f_k}}{\partial x_j}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} u_{i,j}\mathbf{e}_i(\mathbf{z},\mathbf{x})\right) = \Phi\left(\sum_{i=1}^{D} F_{f_k}(\mathbf{z},\mathbf{z}_i)\,\mathbf{e}_i(\mathbf{z},\mathbf{x})\right).$$

Then we have

$$\sum_{i=1}^{D}\sum_{j=1}^{n} -\frac{\frac{\partial R_{f_k}}{\partial z_{i,j}}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} u_{i,j}\mathbf{w}_i = \sum_{i=1}^{D} F_{f_k}(\mathbf{z},\mathbf{z}_i)\,\mathbf{w}_i.$$

So for each $i \in \{1, \ldots, D\}$ we have the following system

$$\left\{ \begin{array}{c} \sum_{j=1}^{n} -\frac{\frac{\partial R_{f_k}}{\partial x_j}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} u_{i,j} = F_{f_k}(\mathbf{z},\mathbf{z}_i) \\ \forall k \in \{1, \ldots, n\} \end{array} \right. \tag{20}$$

Then for each $i \in \{1, \ldots, D\}$ we define

$$\Delta_{\mathbf{z}_i}\overrightarrow{\mathcal{F}}(\mathbf{z}) = \begin{pmatrix} -\frac{\frac{\partial R_{f_1}}{\partial x_1}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} & \cdots & -\frac{\frac{\partial R_{f_1}}{\partial x_n}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} \\ \vdots & \ddots & \vdots \\ -\frac{\frac{\partial R_{f_n}}{\partial x_1}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} & \cdots & -\frac{\frac{\partial R_{f_n}}{\partial x_n}(\mathbf{z},\mathbf{z}_i)}{\mathbf{v}_{\mathbf{z},E}} \end{pmatrix}$$

Following these definitions and since $F_{f_k}(\mathbf{z}, \mathbf{z}_i) = -P_{f_k}(\mathbf{z}_i)$, the system (20) can be written

$$\begin{pmatrix} u_{i,1} \\ \vdots \\ u_{i,n} \end{pmatrix} = \Delta_{\mathbf{z}_i} \overrightarrow{\mathcal{F}}(\mathbf{z})^{-1} \begin{pmatrix} P_{f_1}(\mathbf{z}, \mathbf{z}_i) \\ \vdots \\ P_{f_n}(\mathbf{z}, \mathbf{z}_i) \end{pmatrix}$$

The formula of the theorem is proved. The convergence of the method come from the facts that this method is equivalent to a Newton method and that all roots assumed to be simples. □

REMARK 5.4: *If there is only one root to the system, then the formula 19 is the Newton's iteration formula, and if there is only one variable, the formula 19 is the formula of the classical Weiertrass's method.*

5.2. Algorithm and arithmetic complexity

Recall that we assume to know a monomial basis of the algebra $\mathcal{A}$ as $\mathbb{K}$-vector space and the normal forms in this basis of n polynomials that are not in this basis. We use the notations of the preceding subsections.

5.2.1. Algorithm

The starting points are given thanks to a matrix $\mathbf{z}$ that has its column equal to the vector of the coordinates of the starting points.

ALGORITHM 5.5 (WEIERSTRASS'S ITERATION):

- INPUT *The matrix* $\mathbf{z}$ *formed with the coordinates of the initial points.*
- STEP 1 *We compute the matrices (specialized at* $\mathbf{z}$*)*

$$MV_E(\mathbf{z}) = \begin{pmatrix} \mathbf{z}_1^{\alpha_1} & \dots & \mathbf{z}_1^{\alpha_D} \\ \vdots & \ddots & \vdots \\ \mathbf{z}_D^{\alpha_1} & \dots & \mathbf{z}_D^{\alpha_D} \end{pmatrix}$$

$$MR_{f_k}(\mathbf{z}, \mathbf{x}) = \begin{pmatrix} \mathbf{z}_1^{\alpha_1} & \dots & \mathbf{z}_1^{\alpha_D} & f_1(\mathbf{z}_1) \\ \vdots & \ddots & \vdots & \vdots \\ \mathbf{z}_D^{\alpha_1} & \dots & \mathbf{z}_D^{\alpha_D} & f_n(\mathbf{z}_D) \end{pmatrix}$$

for $k = 1, \cdots, n$. *After we compute the rows:*

$$dv_{f_k}^{(i)}(\mathbf{x}) = \left(\frac{\partial}{\partial x_i} x^{\alpha_1}, \dots, \frac{\partial}{\partial x_i} x^{\alpha_D}, \frac{\partial}{\partial x_i} f_k(\mathbf{x}) \right)$$

for $i \in \{1, \cdots, n\}$ *and* $k \in \{1, \cdots, n\}$.
- STEP 2 *Make the LU-decomposition of* $MV_E(\mathbf{z})$. *Use it to compute the kernels of the matrices* MR_{f_k}. *This gives us the vectors* $r_{f_k} \in \mathbb{K}^{D+1}$, *for each* $i \in \{1, \cdots, n\}$. *If the dimension of the kernel of* $MV_E(\mathbf{z})$ *does not vanish, stop the algorithm with an error setting.*

- STEP 3 *For each* $i \in \{1, \cdots, D\}$, *we construct the matrices* $\Delta_{\mathbf{z}_i}\overrightarrow{\mathcal{F}}(\mathbf{z})$ *by computing their coefficients as follows: we specialize* $dv_{f_k}^{(i)}$ *at* $\mathbf{z}_i$, *compute its product with* $r_{\beta_k}(\mathbf{z})$ *and normalize it by its last coefficient (that is the value of the pseudo Vandermonde determinant), for each* $k \in \{1, \cdots, n\}$. *Then we obtain:* $\Delta_{\mathbf{z}_i}\overrightarrow{\mathcal{F}}(\mathbf{z})$.
- STEP 4 *For each* $i \in \{1, \cdots, D\}$ *we solve the system*

$$(P_{f_1}(\mathbf{z}_i), \cdots, P_{f_n}(\mathbf{z}_i))^t = \Delta_{\mathbf{z}_i}\overrightarrow{\mathcal{F}}(\mathbf{z})\,\mathbf{z}'_i,$$

where $\mathbf{z}'_i$ *is the column indexed by* i *of the output matrix.*
- OUTPUT *The matrix* $\mathbf{z}'$.

5.2.2. Arithmetic complexity for an iteration

PROPOSITION 5.6: *The arithmetic complexity of an iteration of the algorithm (5.5) is bounded by* $\mathcal{O}(D^3 + n^2D^2 + Dn^3)$ *arithmetic operations.*

Proof: We detail here the cost of the different steps:

- In step 1: We begin by making n specialization of $D \times (D+1)$, this gives us $\mathcal{O}(nD^2)$.
- In step 2 and 3, we compute the LU-decomposition the matrix $MV_E(\mathbf{z})$, this gives a complexity $\mathcal{O}(D^3)$. Finally we use this decomposition to compute the kernels of the n matrices MR_{f_k}, this cost $\mathcal{O}(n * D^2)$ arithmetic operations. We specialize nD vectors of length $D+1$, this gives a $\mathcal{O}(nD^2)$ complexity. After we compute n^2D inner products of vectors of length $D+1$, we obtain a complexity term $\mathcal{O}(n^2D^2)$.
- In step 4: We specialize D vectors of length n, this gives us a complexity $\mathcal{O}(nD)$. We solve D linear systems of size $n \times n$, this costs $\mathcal{O}(Dn^3)$ arithmetic operations. Finally we compute n products of $n \times n$ matrices by vectors and we have a complexity $\mathcal{O}(n^2)$. If we denote by L the maximal cost of the evaluation of a P_{f_k} there is also a contribution in $\mathcal{O}(nL)$ arithmetic operations, but this contribution is marginal.

With all these elements we obtain a $\mathcal{O}(D^3 + n^2D^2 + Dn^3)$ arithmetic complexity. □

REMARK 5.7: *In step 2, we do not use the pseudo Vandermonde's structure of the matrices* $MV_E(\mathbf{z})$. *However the use of this structure could give a better complexity. We refer to (26) for the use of the matrix structure for solving systems.*

5.3. Weierstrass's iteration with continuation

Hereafter we use the Weierstrass's iteration function as "predictor-corrector" in a path following method (see (2)). We propose a analogous of a "global Newton's method" to approximate all (or a subset of) the roots of an algebraic system. The idea is the same than the one of the locale Weierstrass's method. We deform an well know algebraic structure to the one we want to describe. The method proposed here is derived from the work exposed in (12) and (30). In those paper the complexity of the global Newton's method is study in the BCSS model approach (see (9)).

Let $\mathbf{z}^{(0)} \in (\mathbb{K}^n)^D$ such as the quotient algebra is defined by n relations $R_{f_1}\left(\mathbf{z}^{(0)}, \mathbf{x}\right), \cdots, R_{f_n}\left(\mathbf{z}^{(0)}, \mathbf{x}\right)$, i.e., if we denote $\mathcal{I}$ the ideal generated by those relations, and by $\mathcal{A}_{\mathbf{z}^{(0)}}$the coordinates algebra $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}$. We assume that $\mathcal{A}_{\mathbf{z}^{(0)}}$ is isomorphic to $\mathbb{K}[x_1, \ldots, x_n]_E$ as a $\mathbb{K}$-vector space. The aim is to deform the system defined by those relations into the initial system.

Definition 5.8: *We define the following application :*

$$\mathcal{H} : \begin{cases} [0,1] \times (\mathbb{K}^n)^D \rightarrow \mathbb{K}[x_1, \ldots, x_n]^n \\ (t, \mathbf{z}) \mapsto \mathcal{H}(t, \mathbf{z}) = \begin{pmatrix} t f_1(\mathbf{x}) + (1-t) R_{f_1}(\mathbf{z}, \mathbf{x}) \\ \vdots \\ t f_n(\mathbf{x}) + (1-t) R_{f_n}(\mathbf{z}, \mathbf{x}) \end{pmatrix} \end{cases}$$

Remark 5.9: *For all $i \in \{1, \cdots, D\}$, $\mathbf{z}_i^{(0)}$ is a solution of $\mathcal{H}\big(0, \mathbf{z}^{(0)}\big) = 0$. Then for Δt small enough, each point $\mathbf{z}_i$ ($i \in \{1, \cdots, D\}$) must be an approximation of a solution of the system $\mathcal{H}\big(\Delta t, \mathbf{z}^{(0)}\big) = 0$.*

The idea of this algorithm is to use this remark in order to apply the multivariate Weierstrass's iteration function to $\mathcal{H}\big(\Delta t, \mathbf{z}^{(0)}\big) = 0$. Then obtain an approximation of the solution of this system, denoted $\mathbf{z}^{(1)}$. So $\mathbf{z}^{(1)}$ must be an approximation of the solution of $\mathcal{H}\big(2\Delta t, \mathbf{z}^{(0)}\big) = 0$, and so we can apply the multivariate Weierstrass's iteration again, and so on. Finally we will get approximation of the solutions of $\mathcal{H}\big(1, \mathbf{z}^{(0)}\big) = 0$, that is to say an approximation of the solutions we sought. In that method, Δt must be small enough. We suppose we have such a value for Δt with $\frac{1}{\Delta t} \in \mathbb{N}$.

Remark 5.10: *We denote by $\mathcal{I}^{(i)}$ the ideal generated by $j\Delta t f_j(x) + (1 - j\Delta t) R_{f_j}\left(\mathbf{z}^{(0)}, \mathbf{x}\right)$, for $j \in \{1, \cdots, n\}$. We assume that at each step $\mathbb{K}[x_1, \ldots, x_n]/\mathcal{I}^{(i)}$ is isomorphic to $\mathbb{K}[x_1, \ldots, x_n]_E$ as a $\mathbb{K}$-vector space. Then $\mathcal{H}$ is an application from $[0,1] \times (\mathbb{K}^n)^D$ to $\mathbb{K}[x_1, \ldots, x_n]_E^n$.*

Under this assumption we have the following algorithm :

Algorithm 5.11:

- Input : Δt, $\mathcal{H}\big(t, \mathbf{z}^{(0)}\big)$.
- *For i from 1 to* $\frac{1}{\Delta t}$ *do*

$$\mathbf{z}^{(i)} = Weierstrass\left(\mathcal{H}\big(i\Delta t, \mathbf{z}^{(0)}\big), \mathbf{z}^{(i-1)}\right)$$

- *Return* $\mathbf{z}^{\left(\frac{1}{\Delta t}\right)}$.

Where the function $Weierstrass$ is the iteration function of the multivariate Weierstrass's method of the previous algorithm.

Remarks 5.12:

- *If we have only one root, this algorithm is the Newton's method with continuation, just as defined in (30).*
- *The singular locus for this methods is given by the vanishing locus of the Vandermonde determinant. This is a complex algebraic variety of codimension 1 in complex space, i.e. of codimension 2 as a real variety. The continuation method use a one dimension real path. Then we can always perturb the system to avoid the singular locus.*

6. Experiences

6.1. Implementation

We have implemented those algorithms in C++ using the tools available in the ALP library. The library ALP is a set of C++ template functions for scientific computation with polynomial and linear algebra. All our functions only use double float arithmetic (or extensions of this arithmetic). But adaptations to several other arithmetics are possible.

To make possible experimentations, we use several programs and language. A set of MAPLE functions allows to create input files for the C++ program. We can call the C++ program directly in bash mode or by using a MAPLE interface function. Our C++ program need in input a file containing the monomial basis, the initial points and the the equations of the system we want to solve. The output is a file containing the points found by the algorithm. The possibility to generate instance with MAPLE and the C++ efficiency made possible a wide exploration of the algorithm's possibilities.

The continuation algorithm begin by a step of interpolation. The initial system is generated by constructing an algebraic system with the wanted structure (quotient algebra is isomorphic as $\mathbb{K}$-vector space to the one of the system we want to solve). This is done by interpolation using relation polynomials from random points. The interpolation process is very efficient and does not limit the method. So we focus on the approximation step. Furthermore, for implementation reason, we are particularly interesting on the use of the double float arithmetic. We just make few remarks to the use of extended arithmetics. The symbol ∞ is used when the results are not significant.

6.2. Numerical experiences of Weierstrass's method

The aim is to study the impact of the two following parameters:

- the number of the steps made during the continuation (denoted nbstep hereafter)
- the number of Weierstrass's iterations made at each step of the continuation (denoted nbiter hereafter)

The experiences reveals other basic facts on the behaviour of the proposed methods. The time is given in second and the accuracy (denoted prec) is evaluate by the norm of the vector form by the evaluations of polynomials of the system evaluated at the approximations given by algorithm (backward error).

6.2.1. Intersections of quadrics

We compute an approximations of the 8 roots of a system consisting in 3 generic quadrics in 3 variables.

nbiter	10		7		5		3	
nbstep	time	prec	time	prec	time	prec	time	prec
55	3	10^{-21}	2	10^{-22}	< 1	10^{-18}	< 1	10
60	3	10^{-21}	2	10^{-22}	1	10^{-20}	< 1	1
75	4	10^{-22}	3	10^{-22}	2	10^{-22}	1	10^{-1}
100	5	10^{-22}	4	10^{-22}	3	10^{-22}	1	10^{-2}
500							8	10^{-11}
1000							15	10^{-15}
2000							31	10^{-19}

We compute an approximations of the 24 roots of a system consisting in 4 generic quadrics in 4 variables. We treat two different problems to illustrate the impact of the conditioning of the system.

nbiter	5		3	
First problem.				
nbstep	time	prec	time	prec
300	132	10^{-22}	80	10^{-12}
250	110	10^{-22}	66	10^{-11}
nbiter	5		4	
Second problem.				
200	407	10^{-7}	326	10^{-1}
300	604	10^{-9}	537	10^{-2}
400	604	10^{-9}	537	10^{-2}

We remark that we do not lose many accuracy when we reduce the number of step, but for instance here we need to keep an accuracy of 10^{-10} at least to converge globally. So we are not able to make less than 250 step in the considered problem.

6.2.2. Intersection of a cubic with quadrics

We compute approximations of the 12 roots of a system consisting in 2 generic quadrics and a cubic in 3 variables. We treat 2 different exemples with different conditionning of the roots.

nbiter	10		7		5		3	
nbstep	time	prec	time	prec	time	prec	time	prec
First problem.								
10	5	10^{-17}	3	10^{-17}	∞	∞	∞	∞
25	4	10^{-17}	3	10^{-17}	∞	∞	∞	∞
50	9	10^{-17}	6	10^{-17}	9	10^{-16}	∞	∞
100	18	10^{-18}	13	10^{-18}	9	10^{-18}	5	10^{-2}
200			25	10^{-17}	18	10^{-18}	11	10^{-7}
300							17	10^{-17}
Second problem.								
20	4	10^{-19}	3	10^{-20}	2	10^{-18}	∞	∞
30	5	10^{-20}	4	10^{-20}	2	10^{-19}	< 1	10^{-2}
50					2		3	10^{-5}

We compute approximations of the 24 roots of a system consisting in 3 quadrics and a cubic in 4 variables.

nbiter	5		4	
200	407	10^{-7}	326	10^{-1}
300	604	10^{-9}	537	10^{-2}
400	814	10^{-22}	645	10^{-4}

6.2.3. Intersections of 3 *cubics*

We compute the an approximations of the 27 roots of a system consisting in 3 generic cubics in 3 variables. We treat an algebraic set include in a disc centred on the origine and with a radius of 10 to controle the size of the Vandermonde determinant. The method is sometimes faster than with less roots because in the previous tests, the roots can have very big and very little coordinates. Here the problem have good properties.

nbiter	10		7		5		3	
nbstep	time	prec	time	prec	time	prec	time	prec
10	10	10^{-14}	4	10^{-14}	1	10^{-1}	∞	∞
20	28	10^{-17}	9	10^{-15}	2	10^{-4}	∞	∞
30	53	10^{-17}	23	10^{-17}	5	10^{-17}	< 1	10^{-8}

6.3. Others experiments

We also treat systems formed by 3 cubics and a quadric (54 solutions) and some particular instance of the problem of the Stewart's platform. But we reach here the limits of the method using only double float arithmetic. For instance, for the Stewart' platforms we compute all roots. The real roots are good conditioned but some of the complex roots are huge and the Vandermonde determinant overflows the arithmetic.

A first implementation with extended was done and allows us to treat bigger problems and need to be improved.

6.4. Conclusions of experiments

The experiments show the efficiency of the proposed method. Some great improvements are possibles. We propose, hereafter, three simple improvements:

- The same number of iterations is used at each step of the continuation process. Often we do not need as many accuracy. We can adapt the algorithm to keep an given accuracy during the path following by changing the number of iterations. The quadratic convergence of the local method can be used at the end of the continuation to increase the accuracy of the final approximation.
- The size the step is always the same during the continuation. We can try to adapt the size of the step. The needed size of the step seem to be conditioned by the distance the singularity.
- In the same way we can try to extend arithmetic when it is needed and to make perturbation and improvements illed conditioned iteration points.

An other possible improvement is to change the representation of the quotient algebra. In the present work, we use a generalization of the Lagrange's polynomials by using idempotents. We can imagine to use a generalization of the Newton's polynomials to have better numerical properties.

The results exposed here and the possibility to make improvements show the interest of the proposed method.

7. Conclusion

In this paper we describe an algebraic framework for the representation of zero-dimensional complete intersection algebraic systems. We give some interpolation formulae that allow us to develop an iterative method to approximate simultaneaously all roots of an algebraic system. We realise experiments that show the efficiency of the method.

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References

[1] Aberth, O. Iteration methods for finding all zeros of a polynomial simultaneously. *Mathematics of Computation 27* (1973).

[2] Allgower, E. L., and Georg, K. *Numerical Path Following.* Springer, 1990.

[3] Atiyah, M., and MacDonald, I. *Introduction to Commutative Algebra.* Addison-Wesley, 1969.

[4] Bellido, A.-M. Construction de fonction d'itération pour le calcul simultané des racines d'une équation $f(s) = 0$. *C.R. Acad. Sci. Paris* (1992).

[5] Bellido, A.-M. *Construction de fonctions d'itération pour le calcul simultané des solutions d'équations et de systèmes d'équations algébriques.* PhD thesis, Université Paul Sabatier de Toulouse, 1992.

[6] Bellido, A.-M. Construction of iteration functions for the simultaneous computation of the solutions of equations and algebraic systems. *Numerical Algorithms 6* (1994), 313–351.

[7] Bini, D. Numerical computation of polynomial zeros by means of aberth's method. *Numerical Algorithms 13* (1996), 179–200.

[8] Blum, L., Crucker, F., Shub, M., and Smale, S. *Complexity and Real Computation.* Springer, 1998.

[9] Blum, L., Shub, M., and Smale, S. On a theory of computation and complexity over the real numbers: Np-compleness, recursive functions and universal machines. *Bulletin of the American Mathematical Society 21*, 1 (July 1989), 1–46.

[10] Cox, D., Little, J., and O'Shea, D. *Ideals, Variety and Algorithms.* Springer-Verlag, New York, 1996.

[11] de Boor, C., and Ron, A. On multivariate polynomial interpolation. *Constructive Approximation 6* (1990), 287–302.

[12] Dedieu, J.-P., and Smale, S. Some lower bounds for the complexity of continuation methods. *Journal of Complexity 14* (1998), 454–465.

[13] Durand, E. *Solution numérique des équations algébriques*, vol. 1. 1968.

[14] Elkadi, M., and Mourrain, B. Géométrie algébrique effective en dimension zéro. Notes de cours, prépublication.

[15] Emsalem, J. Géométrie des points épais. *Bull. Soc. Math. France 106* (1978), 399–416.

[16] Faugére, J.-C. A new efficient algorithm for computing gröbner bases (f_4). *Journal of Pure and Applied Algebra 139* (1999), 61–88.

[17] Fortune, S. Polynomial root finding using iterated eigenvalue computation. In *Proc. ISSAC* (2001), ACM press.

[18] Fromer, A. A unified approach to methods for the simultaneous computation of all zeros of generalized polynomials. *Numer. Math. 54* (1988), 105–116.

[19] Garsia, A., and Haiman, M. *Orbit Harmonics and Graded Representations.* Les éditions du Lacim.

[20] Huber, B., and Sturmfelds, B. A polyhedral method for solving sparce polynomial systems. *Math. Comp. 64*, 212 (1995), 1542–1555.

[21] Kerner, I. Ein Gesamtschrittverfahrenzur Berechnung der Nullstellen von Polynomen. *Numer. Math. 8* (1966), 290–294.

[22] Lascoux, A. Note on interpolation in one and several variables. http://schubert.univ-mlv.fr/ al/.

[23] Li, T. Numerical solution of multivariate polynomial systems by homotopy continuation methods. *Acta Numerica* (1997), 399–436.

[24] Lorentz, G., and Lorentz, R. Bivariate hermite interpolation and application to algebraic geometry. *Numerish Mathematik* (1990).

[25] Mourrain, B. Isolated points,duality and residues. *Journal of Pure and Applied Algebra 117 & 118* (1996), 469–493. Special issue for the Proc. of the 4th Int. Symp. on Effective Methods in Algebraic Geometry (MEGA).

[26] Mourrain, B., and Pan, V. Y. Multivariate polynomials, duality and structured matrices. *Journal of complexity 16*, 1 (2000), 110–180.

[27] Mourrain, B., and Trebuchet, P. Solving projective complete intersection faster. In *Proc. ISSAC* (2000), pp. 231–238.

[28] Pan, V. Solving a polynomial equation: Some history and recent progress. *SIAM Rev. 39*, 2 (1997), 187–220.

[29] Ruatta, O. A multivariate weierstrass iterative rootfinder. In *issac* (London, Ontario, 2001), ACM press.

[30] Smale, S. Complexity theory and numerical analysis. *Acta Numerica* (1997), 523–551.

[31] Vasconcelos, W. v. *Computational Methods in Commutative Algebra and Algebraic Geometry*, vol. 2 of *Algorithms and Computation in Mathematics.* Springer-Verlag, 1998.

[32] Verschelde, J., Verlinden, K., and Cools, R. Homotopies exploiting newton polytopes for solving sparce polynomial systems. *SIAM J. Numerical Analysis 31*, 3 (1994), 915–930.

[33] Weierstrass, K. Neuer Beweis des Satzes, dass Jede Ganze Rationale Function einer Veränderlichen Dargestellt werden kann als ein Product aus Linearen Functionen derselben Veränderlichen. *Mathematische Werke* (1903). Tome 3.

[34] Yakoubsohn, J.-C. Simultaneous computation of all the zero-clusters of a univariate polynomial. Submitted, 2000.